Problem Set 1
Weighted Automata 2020

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Problem 1. Given an NFA $\mathcal{A}$, give a weighted automaton over Natural semiring which outputs the number of accepting paths for a word w on $\mathcal{A}$.

Problem 2. Give a $\mathbb{Z}$-automata $\mathcal{A}$, such that $[\![\mathcal{A}]\!](w) = |w|_a - |w|_b$.

Problem 3. Which of the following are semirings? Argue why.

1. $(\mathbb{N}, \max, +, 0, 0)$
2. $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$
3. $(2^{\Sigma^*}, \cup, \cap, \emptyset, \Sigma^*)$
4. $(2^{\Sigma^*}, \cap, \cup, \Sigma^*, \emptyset)$

*5. $(\mathbb{R} \cup \{-\infty, +\infty\}, \oplus_{\log}, +, +\infty, 0)$, where $x \oplus_{\log} y = -\log(e^{-x} + e^{-y})$

Problem 4. A semiring S is said to be idempotent if $x \oplus x = x, \forall x \in S$. Among the semirings that you have seen in class and the ones in question 3, which semirings are idempotent?

Problem 5. Show that the $n \times n$ matrices over a semiring S with matrix addition and multiplication forms a semiring.

Problem 6. Construct a weighted automaton over Natural semiring that takes a word $w = a^n$ from a unary alphabet $\{a\}$ and outputs n^2. Is it possible to construct such a weighted automata over $(\max, +)$ semiring? Argue why.

Problem 7. Construct a weighted automaton $\mathcal{A}$ over $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \Sigma^*)$, which takes a word w from the alphabet $\Sigma = \{a, b\}$ and outputs a language $L_w \subseteq \{1\}^*$, such that $1^n \in L_w$ iff the n length prefix $w_{\leq n}$ ends with b. For example, $[\![\mathcal{A}]\!](abb) = \{11, 111\}$.

Problem 8. Construct a weighted automaton $\mathcal{A}$ over Reals, which takes a word $w \in \{0, 1\}$, such that $[\![\mathcal{A}]\!](w) = (0.w)_2$ in decimal. For example, $[\![\mathcal{A}]\!](1) = (0.1)_2 = (0.5)_{10}$, and $[\![\mathcal{A}]\!](101) = (0.101)_2 = (0.625)_{10}$.*

*To convert 0.w in binary to decimal, you need to add the reciprocals of powers of two(e.g., $1/2, 1/4, 1/8, 1/16$, for the first, second, third and fourth decimal place, respectively). For example, $0.011 = \frac{1}{2^2} + \frac{1}{2^3} = \frac{1}{4} + \frac{1}{8} = 0.375$.