LECTURE - 8

Last time, we saw Distance Automata (word weights are distances).

- Word Evaluation
  - linear in $|W|$
  - cubic in number of states
- Support Language
  - Always regular
- Threshold Problem } undecidable
- $A \bowtie B$ } undecidable
- Boundedness Problem
  - Are the weights produced by this automaton bounded by some value?
  - + Equivalence modulo boundedness
  - ~~For any two bounded subsets of Σ^*, is the distance of each word equal to the distance~~
  - + Cost Functions } The mathematically oriented reader should look these up
  - + B-automata } The mathematically oriented reader should look these up

Probabilistic Automata

Weighted Automata over $(\mathbb{R}, +, \times)$ with syntactic restrictions;

As before, a weighted automaton $\mathcal{A}$ with $\langle \lambda, \mu, \gamma \rangle$ with restrictions

- all entries in λ or μ or γ come from $[0, 1]$
- $\sum_i \lambda_i = 1$
- $\sum_j \mu^a_{ij} = 1 \quad \forall i \ \forall a \in \Sigma$ — (ii)
- $\gamma_i \in \{0, 1\} \quad \forall i$

These are also called probabilistic reactive systems.

A variation where words are stochastically generated, instead of being fed by the environment is known as generative probabilistic automata. (We will not look at this model)

Condition (ii) above is often stated as $\mu(a)$ being a stochastic matrix, i.e, a matrix whose row sums are 1.

With the definition of $\mu(w)$ as before, we claim that $\mu(w)$ is always a stochastic vector for all $w \in \Sigma^*$.

To verify this, we first check that the product of two stochastic matrices is stochastic.

If (a_{ij}) and (b_{ij}) are stochastic matrices, and (c_{ij}) is their product, then

$$\sum_j c_{ij} = \sum_j \left(\sum_k a_{ik} b_{kj} \right) = \sum_k \sum_j a_{ik} b_{kj}$$

$$= \sum_k a_{ik} \left(\sum_j b_{kj} \right) = \sum_k a_{ik} = 1$$

as desired.

Similarly, we can show that the product of a stochastic vector with a stochastic matrix is also a stochastic vector.

Some Closure Properties

1. If $\mathcal{A}$ is a probabilistic automaton, then there exists another probabilistic automaton $\mathcal{B}$ s.t $[\![\mathcal{B}]\!]$ realises the function $1 - [\![\mathcal{A}]\!]$

   Proof If $\mathcal{A} = \langle \lambda, \mu, \gamma \rangle$, consider $\mathcal{B} = \langle \lambda, \mu, \bar{\gamma} \rangle$ where $\bar{\gamma}_i = 1 - \gamma_i$ $\square$

2. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are probabilistic automata, there is a probabilistic automata $\mathcal{B}$ s.t

   $[\![\mathcal{B}]\!] = [\![\mathcal{A}_1]\!] \, [\![\mathcal{A}_2]\!]$, i.e, $\forall w \in \Sigma^*$, $[\![\mathcal{B}]\!](w) = [\![\mathcal{A}_1]\!](w) \times [\![\mathcal{A}_2]\!](w)$

   Proof Idea: Construct a Product Automaton.

   Let $Q(\mathcal{B}) = Q(\mathcal{A}_1) \times Q(\mathcal{A}_2)$

   Then, let $\lambda^{\mathcal{B}}_{(p,q)} = \lambda^{\mathcal{A}_1}_p \cdot \lambda^{\mathcal{A}_2}_q$

   $$\mu^{\mathcal{B}}_{(p,q)\to(r,s)}(a) = \mu^{\mathcal{A}_1}_{p\to r}(a) \cdot \mu^{\mathcal{A}_2}_{q\to s}(a)$$

   $$\gamma^{\mathcal{B}}_{(p,q)} = \gamma^{\mathcal{A}_1}_p \cdot \gamma^{\mathcal{A}_2}_q$$ (This is intuitively like the $\wedge$ of the two values)

$\square$

3. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are probabilistic automata, and α, β are in $[0, 1]$ with $\alpha + \beta \leq 1$

Then, there exists $\mathcal{B}$ which realises the function $\alpha [\![\mathcal{A}_1]\!] + \beta [\![\mathcal{A}_2]\!]$

<u>Proof</u>

Let $\perp$ be an extra state not in $\mathcal{A}_1 \cup \mathcal{A}_2$

Construct an automaton $\mathcal{B}$ by taking the disjoint union of scaled copies $\alpha\mathcal{A}_1$ and $\beta\mathcal{A}_2$ as follows:

$$\lambda = [\ \underbrace{\qquad\qquad}_{\alpha\,\lambda_{\mathcal{A}}}\ \underbrace{\qquad\qquad}_{\beta\,\lambda_{\mathcal{A}_2}}\ \underbrace{\quad}_{1-\alpha-\beta}\ ]$$

$$\mu(a) = \begin{bmatrix} \boxed{\cancel{\alpha}\ \mu^{\mathcal{A}_1}(a)} & & \\ & \boxed{\cancel{\beta}\ \mu^{\mathcal{A}_2}(a)} & \\ & & \boxed{1} \end{bmatrix}$$

$$\gamma = [\ \underbrace{\qquad\qquad}_{\gamma_{\mathcal{A}_1}}\ \underbrace{\qquad\qquad}_{\gamma_{\mathcal{A}_2}}\ 0\ ]$$

□

4. If $\mathcal{A}$ is a probabilistic automata, then we can construct $\mathcal{B}$ such that $[\![\mathcal{B}]\!](w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ [\![\mathcal{A}]\!](w) & \text{o/w} \end{cases}$

<u>Proof</u> Add an extra state T

$$\lambda = \begin{array}{cc} T & Q \\ [\ 1 & \overbrace{\ldots 0000000 \ldots}\ ] \end{array}$$

$$\mu(a) = \begin{array}{c|cc} & T & Q \\ \hline T & 0 & [\ \lambda^{\mathcal{A}} \cdot \mu^{\mathcal{A}}(a)\ ] \\ Q & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \mu^{\mathcal{A}}(a) \end{array}$$

$$\gamma = (0\ \underbrace{\qquad\qquad}_{\gamma^{\mathcal{A}}})$$

□

<u>Probabilistic Language</u>

$\mathcal{L} \subseteq \Sigma^*$ is a probabilistic language iff there exists ~~an~~ probabilistic automata $\mathcal{A}$ and an $\alpha \in [0, 1]$ s.t $\mathcal{L} = \{ w \mid [\![\mathcal{A}]\!](w) > \alpha \}$.

We shall ~~revisit~~ see undecidability results about probabilistic languages next time.