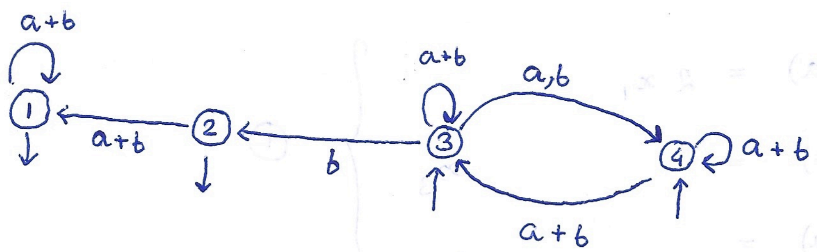
LECTURE - 5

Recall: We defined Reachable Vectors $\lambda\mu(\Sigma^*)$ last time and gave an algorithm to compute a basis for $\langle \lambda\mu(\Sigma^*)\rangle$

Apriori we have that $\langle \lambda\mu(\Sigma^*)\rangle \subseteq S^n$, but we may be able to find a basis $\{x_1, \dots x_k\}$ for $\langle\lambda\mu(\Sigma^*)\rangle$ with $k \leq n$. In other words, the set of reachable vectors may be obtained as a k dimensional subspace of S^n. In such a case, we should be able to represent the given automaton ~~as same~~ with k states. Indeed, this is what we shall argue. (We assume throughout that S is a field.)

Example

$\mathcal{A}$ =



$$\lambda = [0 \quad 0 \quad 1 \quad 1]$$

$$\mu(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

By running the algorithm from LECTURE - 4 with minor modifications, we see that the following is indeed a basis for $\langle\lambda\mu(\Sigma^*)\rangle$:

$$\text{Basis} = \{x_1, x_2, x_3\}$$

$x_1 = \lambda = (0\ 0\ 1\ 1)$

$x_2 = (0\ 1\ 0\ 0)$

$x_3 = (1\ 0\ 0\ 0)$

Define $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

The weighted automaton may be understood as an object that assigns with each word $w \in \Sigma^*$, the vector $\lambda \mu(w) \in S^n$. However, if we indeed find that $\langle \lambda \mu(w) \rangle \simeq S^k$, for some $k < n$, then we may represent each $\lambda \mu(w) \in S^n$ via $\lambda' \mu'(w) \in S^k$ such that there is a (linear) transformation that takes $\lambda' \mu'(w)$ to $\lambda \mu(w)$. This transform happens to be X as we will check.

In summary, if $(\alpha_1 \ \alpha_2 \ \alpha_3)$ is a vector with respect to the new basis, then $(\alpha_1 \ \alpha_2 \ \alpha_3) X$ is the coordinates of the same vector with respect to the standard basis.

~~Suppose that~~

Now, returning to the example, we have that:

$$\left.\begin{aligned} x_1 \mu(a) &= 2x_1 \\ x_2 \mu(a) &= x_3 \\ x_3 \mu(a) &= x_3 \end{aligned}\right\} \ (1)$$

and

$$\left.\begin{aligned} x_1 \mu(b) &= 2x_1 + x_2 \\ x_2 \mu(b) &= x_3 \\ x_3 \mu(b) &= x_3 \end{aligned}\right\} \ (2)$$

So, define
$$\mu'(a) := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mu'(b) := \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that System (1) and System (2) are simply facts that

$$X \mu(a) = \mu'(a) X \qquad \text{and} \qquad X \mu(b) = \mu'(b) X$$

These equations are just manifestations of the fact that we wish the following diagram to commute

$$\begin{tikzcd} S^n \arrow[r, "\mu(a)"] & S^n \\ S^k \arrow[u, "X"] \arrow[r, "\mu'(a)"'] & S^k \arrow[u, "X"'] \end{tikzcd}$$

Its not hard to argue that we will be able to find such a $\mu'(a)$ everytime.

Indeed, suppose $x_1 \in \langle \lambda \mu(\Sigma^*) \rangle$, then there are words $w_1 \dots w_k$ such that $x_1 = \sum_1^k \alpha_i w_i$. Then, we have that $x_1 \mu(a) = (\sum \alpha_i w_i) \mu(a)$

$$= \sum \alpha_i \mu(w_i a) \in \langle \lambda \mu(\Sigma^*) \rangle$$

Since $\{x_1 \dots x_k\}$ is a basis, we should be able to represent $x_1 \mu(a)$ as a linear combination of elements in them. Using a similar argument, we may find λ' s.t. $\lambda' X = \lambda$.

Finally, define $\gamma' := X\gamma$. Again, note that if $(\alpha_1\ \alpha_2\ \alpha_3)$ is a representation of a vector in the new basis, then $(\alpha_1\ \alpha_2\ \alpha_3) X$ is their representation in the standard basis and $((\alpha_1\ \alpha_2\ \alpha_3) X)\gamma = (\alpha_1\ \alpha_2\ \alpha_3)\gamma'$

Thus, we may define $\mathcal{A}'$ whose states are $\{x_1, x_2, x_3\}$ and transitions and other data are given as follows: $\mathcal{A}' = \langle \lambda', \mu', \gamma' \rangle$

Claim For all $w \in \Sigma^*$, $[\![\mathcal{A}]\!](w) = [\![\mathcal{A}']\!](w)$

Proof The claim is equivalent to the fact that $\lambda' \mu'(w) \gamma' = \lambda \mu(w) \gamma$

$$\iff \lambda' \mu'(w) X \gamma = \lambda \mu(w) \gamma$$

We can prove this equivalent statement by induction on w. $\iff \lambda' \mu'(w) X = \lambda \mu(w)$

Base Case If $w = \varepsilon$, Then, we need to check that $\lambda' X = \lambda$

This is true by definition.

<u>Inductive Step</u>

$$\begin{aligned}\lambda' \cdot \mu'(wa) \cdot X &= \lambda' \mu'(w) \mu'(a) X \\ &= \lambda' \mu'(w) X \mu(a) && [\text{definition of } \mu'(a)] \\ &= \lambda \mu(w) \mu(a) && [\text{Induction Hypothesis}] \\ &= \lambda \mu(wa)\end{aligned}$$

A similar argument holds for $\lambda' \cdot \mu'(wb) \cdot X = \lambda \mu(wb)$

QED

We call this automata the <u>Reduced Automata</u>. This Construction is ~~indeed~~ a counterpart to the subset construction, in the sense that we club together different configurations ~~different runs~~ as linear Combinations of a certain basis. Note that the field structure is necessary here since, the coefficients of the reduced automata may contain additive or multiplicative inverses. This is the reason we Cannot do such a construction for NFA, as the "reachable vectors" do not admit a small basis.

<u>Exercise</u> (Not necessarily relevant to the above theory)

Given an NFA $\mathcal{A}$, show that

$$\text{Determenize}(\text{Reverse}(\text{Determenize}(\text{Reverse } \mathcal{A})))$$

is ~~a~~ the minimal DFA for $\mathcal{A}$