LECTURE - 2

Weighted Language - functions of type $\Sigma^* \longrightarrow \mathbb{S}$

$\left[\mathbb{S} = \begin{pmatrix} (S, +, \cdot, 0, 1) \\ \text{- semiring} \end{pmatrix} \right]$

Weighted Automata over $\underset{\text{alphabet}}{\Sigma}$ and $\mathbb{S}$ — semiring

Syntax:

$$\mathcal{A} = (Q, \Delta, \lambda, \gamma, \mu)$$

Q - finite set of "states"

$\Delta \subseteq Q \times \Sigma \times Q$

edge weights $\begin{cases} \mu : \Delta \longrightarrow S \\ \text{[or could directly specify } \Delta \text{ as } Q \times \Sigma \times Q \longrightarrow \mathbb{S} \\ \text{with } 0 \text{ on absent edges]} \end{cases}$

λ - initial weight : $Q \longrightarrow S$

γ - final weight : $Q \longrightarrow S$

Semantics:

Given $w = a_1 a_2 \cdots a_n \in \Sigma^*$

define $$[\![A]\!](w) := \sum_{\substack{q_0 \cdots q_n \\ \in Q^n}} \lambda(q_0) \left(\prod_{i=1}^{n} \mu(q_{i-1}, a_i, q_i) \right) \gamma(q_n)$$

$[\![A]\!] : \Sigma^* \longrightarrow \mathbb{S}$ — this is the weighted language

Exercises

1. Construct an weighted automata that counts the number of a's in a word. Take $\Sigma = \{a, b\}$, $S = (\mathbb{Z}, +, \cdot, 0, 1)$

2. Construct an automata that computes the same function over $(\mathbb{Z} \cup \{+\infty\}, \min, +, +\infty, 0)$

3. " " " " " $w \mapsto |w|_a - |w_b|$ in $(\mathbb{Z}, +, \cdot)$

Evaluation Problem

Input : A over S, and $w \in \Sigma^*$

Output : $[\![A]\!](w)$

First definition is $|Q|^{|w|}$. Redefine the automata using matrices.

Represent A by $\langle \lambda, \mu(a), \mu(b), \gamma \rangle$

where $\lambda, \gamma \in S^{|Q|}$, as before

and $\mu(a)$ is a matrix over S s.t

$\mu(a)_{i,j} = x$ iff there is a transition from i to j on a with wt. x.

Now, extend μ to Σ^* ~~with~~ in the following way:

$$\mu(\varepsilon) = I$$

$$\mu(wa) = \mu(w) \cdot \mu(a) \quad \text{matrix multiplication}$$

$$\mu(wb) = \mu(w) \cdot \mu(b)$$

~~Define~~ With this definition,

<u>Claim</u> $\mu(w)_{pq}$ = Sum of Product of Weights over all Paths from p to q on w

<u>Proof</u> <u>Base Case</u> $\mu(\varepsilon)$, $\mu(a)$, $\mu(b)$

Follows from definition.

<u>Inductive Case</u>

$$\mu(wa) = \mu(w) \cdot \mu(a)$$

$$\Rightarrow \mu(wa)_{pq} = \sum_r \mu(w)_{pr} \cdot \mu(a)_{rq}$$

$$= \sum_r \left(\text{Sum of weights of } w\text{-paths from } p \text{ to } r\right)\left(\text{Weight of edge from } r \text{ to } q\right) \quad [IH]$$

$$= \sum_r \left(\text{Sum of weights of } wa \text{ paths that pass through } r \text{ on reading } w\right) \quad [\text{distributivity}]$$

$$= \text{Sum of weights of all } wa \text{ paths}$$

$\square$

Intution : $\mu(a)$, $\mu(b)$ are actions that ~~distort a probability distribution~~ transform a weight distribution

λ is an ~~ost~~ initial distribution. γ is an aggregation function.

Observe that for an word $w = a_1 a_2 \ldots a_n$ and an automata as above, we can compute

$$[\![A]\!](w) = \lambda \cdot \mu(w) \cdot \gamma$$

in time $O(|w|\,|Q|^3 \beta)$ by doing matrix multiplication.