LECTURE - 13

<u>Recall</u> from the theory of classical finite state automata theory:

If L is a language and w is a word,

$$w^{-1}L := \{v \mid wv \in L\}$$ called the residual of L on w

Also, say that $u \equiv_L v$ iff $\forall w \in \Sigma^*,\ uw \in L \Leftrightarrow vw \in L$

<u>Myhill - Nerode Theorem</u> L is regular iff $\equiv_L$ has finite index

Or equivalently, iff $\text{Residuals}(L) = \{w^{-1}L \mid w \in \Sigma^*\}$ is finite.

We will develop a weighted counterpart of this theory.

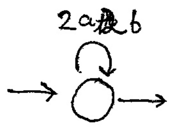
Suppose $f : \Sigma^* \to \mathbb{S}$ is a weighted language, and $w \in \Sigma^*$, then define $w^{-1}f : \Sigma^* \to \mathbb{S}$

$$x \mapsto f(wx)$$

Let $\text{Residuals}(f) = \{w^{-1}f \mid w \in \Sigma^*\}$

Is $\text{Residuals}(f)$ finite when f is recognizable? No.

For Instance, Consider $x \mapsto 2^{|x|}$, which is realised by the following automata



Notice that $w^{-1}f = x \mapsto f(wx) = x \mapsto 2^{|w|+|x|} = x \mapsto 2^{|w|}(2^{|x|})$

For every v such that $|w| \neq |v|$, $w^{-1}f \neq v^{-1}f$. Thus, there are infinitely many residuals.

To find an analogue of Myhill - Nerode, we look for a finite set of objects which generate the set of residuals.

<u>Left $\mathbb{S}$ - modules</u> Suppose that $\mathbb{S}$ is a semiring. A (left) module over $\mathbb{S}$ is an object M endowed with a $+$ - operation, which makes M a commutative monoid, along with a "scalar product" $\times ; \mathbb{S} \times M \to M$ s.t.

- $s(m_1 + m_2) = sm_1 + sm_2$
- $(s_1 + s_2)m = s_1 m + s_2 m$
- $s_1(s_2 m) = (s_1 s_2)m$
- $1_{\mathbb{S}}\, m = m$
- $0_{\mathbb{S}}\, m = 0_M$
- $s\, 0_M = 0_M$

For brevity, we will call such objects modules.

A submodule of a module M is a subset of its elements which are closed under addition and scalar multiplication. Note that a submodule is a module in its own right.

Notice that Series, denoted $\mathbb{S}\langle\langle\Sigma^*\rangle\rangle$ henceforth form a module under the addition of series and standard scalar multiplication.

A module is finitely generated if there is a finite set $\{m_1, \ldots m_n\}$ s.t for every $m \in M$,

$$m = \sum_{i=1}^{n} \alpha_i m_i \quad \text{for some } \alpha_i \in \mathbb{S}$$

We call a submodule M of $\mathbb{S}(\langle\langle\Sigma^*\rangle\rangle)$ stable, if $\forall w \in \Sigma^*$, $f \in M \Rightarrow w^{-1} f \in M$

Notice that the set of polynomials are a stable submodule. Indeed, if ~~w is the non zero word~~ the length of the longest word which has a non-zero coefficient in some f is l, then for any $w \in \Sigma^* \setminus \{\varepsilon\}$, the length of the longest word with non-zero coefficient would be $l - |w|$ or smaller. So, $w^{-1} f$ is also a polynomial.

However, the submodule of polynomials ~~are~~ is not ~~stable~~ finitely generated

(Analogue of Myhill Nerode Theorem for Weighted Automata) Suppose $f : \Sigma^* \longrightarrow \mathbb{S}$.

f is recognizable iff f is a member of a finitely generated stable submodule of $\mathbb{S}\langle\langle\Sigma^*\rangle\rangle$

Proof ($\Leftarrow$) Suppose $M \subseteq \mathbb{S}\langle\langle\Sigma^*\rangle\rangle$ is generated by $\{f_1 \cdots f_n\}$ and $f \in M$.

Then, we can realise f by a finite state weighted automata as follows: Let the set of states be $\{f_1 \ldots f_n\}$. Then, since M is stable and generated by $\{f_1 \ldots f_n\}$, it is possible to pick $\{\mu^a\}_{ij}$ s.t.

$$a^{-1} f_i = \sum_{i=1}^{n} \mu^a_{ij} f_j$$

This gives us the matrices $\mu(a)$ for $a \in \Sigma$.

Now, define $\gamma_i = f_i(\varepsilon)$

and λ_j s.t $f = \sum_{j=1}^{n} \lambda_j f_j$

Then, $\langle \lambda, \mu, \gamma \rangle$ gives the automata we were looking for.

($\Longrightarrow$) Suppose f is recognizable via $\langle \lambda, \mu, \gamma \rangle$

Define f_i to be the function that takes w to $(\mu(w)\gamma)_i$

Then $f = \sum_i \lambda_i f_i$

The submodule we are looking for is the one generated by $\{f_1, f_2 \dots f_n\}$

To show that it is stable, it is enough to show that $a^{-1} f_i$ can be written as $\sum c_{ij} f_j$ for coefficients $c_{ij} \in S$ for every generator f_i and every $a \in \Sigma$. Indeed, by definition, we have that

$$a^{-1} f_i = \sum_{j=1}^{n} \mu(a)_{ij} \, f_j$$

<u>QED</u>

<u>Aside</u> The function $a^n \mapsto 2^{2^n}$ is not ~~ex~~ recognizable over $(\mathbb{N}, +, \times)$ and $\Sigma = \{a\}$

Suppose an automaton $\mathcal{A}$ has $|Q|$ many states with the largest weight k. Then, the ~~largest~~ weight of a single run ~~of~~ ~~an~~ word a^n is bounded by k^n. The number of such runs is bounded by $|Q|^n$. So, $[\![\mathcal{A}]\!](a^n) \leq (|Q|\, k)^n$. For sufficiently large n, $(k|Q|)^n < 2^{2^n}$.

<u>Comments on Hadamard Product</u>

For $f, g : \Sigma^* \to S$, define $f \odot g$ by $w \mapsto f(w) \times g(w)$ where $\times$ is the semiring multiplication. If the semiring is commutative, $f \odot g$ is recognizable whenever f and g are. This follows from a construction similar to the one in LECTURE-8

<u>Exercise</u> Working on the semiring $(\Sigma^*, +, \cdot)$, suppose

$$\left.\begin{aligned} f &: a \mapsto a \\ &\quad b \mapsto \varepsilon \\ g &: a \mapsto a \\ &\quad b \mapsto b \end{aligned}\right\}$$

Show that f and g are recognizable but $f \odot g$ is not.

<u>Aside</u> Weighted Automata over the semiring of languages are transducers. Several important decision problems for transducers are undecidable.