LECTURE-12

Continuing our discussion, we proceed to show that

If f is proper and recognizable, so is f^*

<u>Step I</u> In step I, we make sure that there is at most one non-zero entry in $\lambda^{\mathcal{A}}$ where $\mathcal{A}$ is the automaton that we will construct. Furthermore, there are no non-zero transitions entering the initial state.

To achieve this, let $\mathcal{A} = \langle \lambda, \mu, \gamma \rangle$ be the automaton realizing f. Let Q be the set of states of $\mathcal{A}$. Add ~~an~~ a fresh state q_0 to $\mathcal{A}$ and for every $a \in \Sigma$, $q, q_2 \in Q$, ~~add trans.~~ and transitions $\xrightarrow{w} q \xrightarrow[w, a]{} q_2$, add the transition $q_0 \xrightarrow[w w, a]{} q_2$. Set the in-weights of every state except q_0 to 0. Set the in-weight of q_0 to 1.

<u>Step II</u> Let q_0 be the only initial state of $\mathcal{A}$. Set the out weight of q_0 to 1. Set the out-weight of all other ~~transitions~~ states to 0.

For every set of transitions in the original automaton of the form $q \xrightarrow[aw]{} q_f \xrightarrow[w_0]{}$ add a transition $q \xrightarrow[a w w_0]{} q_0$.

With this, we have shown that Recognizable Languages $\supseteq$ Rational Languages. We turn to prove the converse.

<u>Recognizable Languages are Rational</u>

In standard finite state automata theory, we construct rational expressions by eliminating states. Here, we will use a more algebraic approach.

First, we make some observations.

- $(\Sigma^{Rat}, +, \cdot, 0, 1)$ is a semiring where $+$ is series addition and $\cdot$ is Cauchy Product and 0 and 1 are the 0 and 1 polynomials.
- If S is a semiring, we may consider $M(S)$, matrices over entries in S, with an induced semiring structure, as remarked in LECTURE-1. Therefore, we will consider the Matrices over Rational Series together with their semiring operations.
- Provided that a notion of convergence exists, we may define the $*$ operation on arbitrary semirings $x^* = 1 + x + x^2 + \cdots$. This is indeed the case with $M(\Sigma^{Rat})$

<u>Arden's Lemma for Qualitative Languages</u> If $X, L_1, L_2 \subseteq \Sigma^*$ are languages satisfying $X = L_1 X + L_2$ and $\varepsilon \notin L_1$, then $X = L_1^* L_2$

We show an analogous result for series.

<u>Proposition</u> Suppose f is a series satisfying $f = g \cdot f + h$ where g is proper. Then, $f = g^* h$

<u>Proof</u>

$$f = g \cdot f + h$$
$$= g(g \cdot f + h) + h = g^2 f + g \cdot h + h$$
$$= g^2 (g f + h) + g h + h = g^3 f + g^2 h + g h + h$$

First, by plugging in $f = g^* h$, we see that $g^* h$ is a solution:

$$g(g^* h) + h$$
$$= g g^* h + h$$
$$= g(1 + g + g^2 + \ldots) h + h$$
$$= (gh + g^2 h + g^3 h + \ldots) + h$$
$$= h + gh + g^2 h + \ldots = g^* h$$

To show uniqueness,

$$f = g \cdot f + h$$
$$= g(gf + h) + h = g^2 f + gh + h$$
$$= \ldots = g^{i+1} f + (g^i + g^{i-1} + \ldots + g + 1) h$$

Suppose w is a word of length i which has a non-zero coefficient in f.

Then, w does not have a non-zero coefficient in $g^{i+1} f$, since g is proper.

Thus, w has the same coefficient in $(g^i + g^{i-1} + \ldots + g + 1) h$ as it does in $g^* h$.

We can similarly argue for any word with a non-zero coefficient in $g^* h$.

Thus, all words agree on $g^* h$ and f. Thus, $g^* h$ is an unique solution.

QED

Given the automaton $\langle \lambda, \mu, \gamma \rangle$, we will interpret μ as $\mu(a) + \mu(b)$. This will give us a matrix of series. Similarly, we may think of λ and γ to be vectors of series in a natural way.

Provided we compute μ^* in the manner understood above, the entry $(\mu^*)_{ij}$ would correspond to the function computed by the automata if i was a state with in-weight 1 and j was a state with out-weight 1. Thus, the dot product $\lambda \mu^* \gamma$ would give us the rational expression we require for f.

It remains to compute the entries of μ^* as rational expressions.

Computing μ^*

First, we have that $\mu^* = \mu \cdot \mu^* + \mathbb{1}$ where $\mathbb{1}$ is the identity matrix

~~We suppose that μ~~

But, we may explicitly carry out this multiplication, and get

$$(\mu^*)_{ij} = \left(\sum_k \mu_{ik} (\mu^*)_{kj} \right) + \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This is a set of $|Q| \times |Q|$ equations, one for each i and j.

We can eliminate each $(\mu^*)_{ij}$ one at a time, by applying Arden's Lemma successively.