# LECTURE-10

We continue our discussion of the value-$\alpha$ problem.

Input : $\mathcal{A}$

Qn : Does $\exists w \in \Sigma^*$, $[\![\mathcal{A}]\!](w) = \alpha$ hold?

This is decidable for $\alpha = 0$, as discussed before. By considering $1 - [\![\mathcal{A}]\!]$ using the discussed closure properties, we can also decide this for $\alpha = 1$.

We ~~use~~ a reduction from PCP to show undecidability of ~~solve~~ the value-$\frac{1}{2}$ problem.

First, we describe PCP:

Let $\Sigma$ be a finite set

Input : $f : \Sigma \to \{0,1\}^*$
$g : \Sigma \to \{0,1\}^*$

Output : Does there exist $w \in \Sigma^*$ such that $\hat{f}(w) = \hat{g}(w)$?

where $\hat{f}$ and $\hat{g}$ are the lifts ~~and~~ of $f$ and $g$ to $\Sigma^*$

Consider the following modification of a given PCP instance:

Given $f : \Sigma \to \{0,1\}^*$, let ~~$f'(a_1 a_2 \ldots a_n) =$~~

$$f'(a) = f(a) \gg\!= \begin{Bmatrix} 0 \mapsto 01 \\ 1 \mapsto 11 \end{Bmatrix} \quad \text{for } a \in \Sigma$$

By way of example, if $f(a) = 010$, then $f'(a) = 011101$

Or, with more verbose notation:

if $f(a) = a_1 a_2 \ldots a_n$

then $f'(a) = a_1 1\, a_2 1 \ldots a_n 1$

It is clear that this modification preserves solutions, in the sense that $f$ and $g$ have a solution in original PCP iff ~~they~~ $f'$ and $g'$ do. Thus, we may assume that $\hat{f}(w)$ ends with 1 for any $w \in \Sigma^*$.

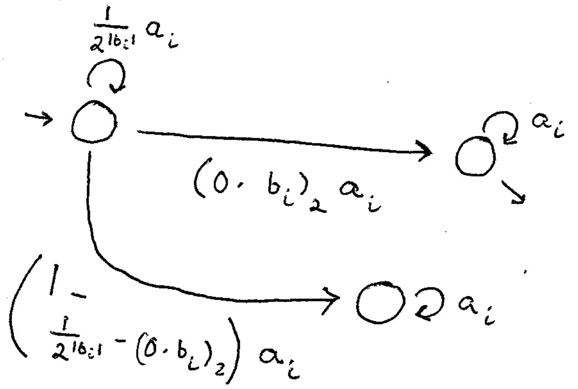
Now, we may note that $(0.\hat{f}(w))_2 = (0.\hat{g}(w))_2$ iff $\hat{f}(w) = \hat{g}(w)$

where $(\ )_2 : \Sigma^* \to [0,1]$

is the function that takes binary strings to the value they denote.

We show that given an $f : \Sigma \to \{0,1\}^*$, the function $w \mapsto (0.f(w))_2$ is realisable by a probabilistic automata.

Suppose $f$ takes the symbol $a_i \in \Sigma$ to the string $b_i \in \{0,1\}^*$. Then, for each such $a_i$, we add transitions in the following way:




We'll call this automata $\mathcal{A}_f$.

Consider the automaton that computes $\frac{1}{2}[\![\mathcal{A}_f]\!] + \frac{1}{2}(1 - [\![\mathcal{A}_g]\!])$.

We claim that this automata has a word of weight $\frac{1}{2}$ iff PCP has a solution.

Indeed, if

$$\frac{1}{2}[\![\mathcal{A}_f]\!](w) + \frac{1}{2}(1 - [\![\mathcal{A}_g]\!](w)) = \frac{1}{2},$$

$$\iff [\![\mathcal{A}_f]\!](w) = [\![\mathcal{A}_g]\!](w)$$

$$\iff (0.\hat{f}(w))_2 = (0.\hat{g}(w))_2$$

$$\iff \hat{f}(w) = \hat{g}(w)$$

Thus, we have:

<u>Proposition</u> The value-$\frac{1}{2}$ problem is undecidable.

Next, we show that value-$\alpha$ is undecidable for $0 < \alpha < 1$.

<u>When $\alpha < \frac{1}{2}$</u>

Consider $\mathcal{A}$ that computes $\alpha [\![\mathcal{A}_f]\!] + \alpha (1 - [\![\mathcal{A}_g]\!])$.

By a similar argument, this automaton has a word of weight $\alpha$ iff $[\![\mathcal{A}_f]\!](w) = [\![\mathcal{A}_g]\!](w)$ for some $w$. And Hence, we can reduce PCP to the value-$\alpha$ problem.

<u>When $\alpha > \frac{1}{2}$</u>

$\mathcal{A}$ has a word of weight $\alpha$ iff $(1 - [\![\mathcal{A}]\!])$ has a word of weight $(1-\alpha)$.

But, $\alpha > \frac{1}{2} \Rightarrow 1 - \alpha < \frac{1}{2}$. So, this reduces to the previous case.

QED

<u>Proposition</u> Given $\mathcal{A}$, it is undecidable to check if there exists $w$ s.t. $[\![\mathcal{A}]\!](w) \geq \alpha$

<u>Proof</u> We reduce the Value-$\frac{1}{2}$ problem to this problem.

Recall that given probabilistic machines $\mathcal{A}$ and $B$, we can find $C$ that computes $[\![\mathcal{A}]\!][\![B]\!]$

We call this the <u>Hadamard Product</u>

Consider the automaton $B$ that computes $([\![A]\!](1-[\![\mathcal{A}]\!]))$.

$$\left([\![\mathcal{A}]\!](1-[\![\mathcal{A}]\!])\right)(w) \leq \frac{1}{4}$$ with equality iff $[\![\mathcal{A}]\!](w) = \frac{1}{2}$

Thus, $[\![\mathcal{A}]\!]$ has a word of weight $\frac{1}{2}$ iff $[\![B]\!]$ has a word of weight $\geq \frac{1}{4}$. ∎

Next, we consider the following Problem

Input: ~~Weighted~~ Probabilistic Automata $\mathcal{A}$, Value $\alpha$

Question: Is $\alpha$ an isolated cut point of $\mathcal{A}$?

We show that this is undecidable.

<u>Proof</u>

We consider the following problem:

Input: Finite Set $\Sigma$

Morphisms $f: \Sigma \to \{0,1\}^*$

$g: \Sigma \to \{0,1\}^*$

s.t. $f(a) \neq g(a)$ for some $a \in \Sigma$

Output: Is the set

$\{$ Longest Common Prefix $(\hat{f}(w), \hat{g}(w)) \mid w \in \Sigma^* \}$ finite?

This problem is undecidable. We omit the proof.

We reduce our problem from this.

Again, let $\mathcal{A}$ be an automata that computes $\alpha [\![\mathcal{A}_f]\!] + \alpha(1 - [\![\mathcal{A}_g]\!])$ as before.

If the set of longest common prefix is infinite,

given any $\varepsilon > 0$, there is $w$ such that $|(0.f(w))_2 - (0.g(w))_2| < \varepsilon$

and thus $|[\![\mathcal{A}]\!](w) - \alpha| < \varepsilon$

This can be witnessed by $wa$ such that $f(a) \neq g(a)$ but $f(w)$ and $g(w)$ have a sufficiently long common prefix

On the other hand, if we have $|[\![\mathcal{A}]\!](w) - \alpha| < \varepsilon$ for $\varepsilon < 2^{i}$, the first $i$ bits of $f(w)$ and $g(w)$ must agree. Thus, the set of longest common prefixes is infinite iff $\mathcal{A}$ has $\alpha$ as a cluster point.

$\blacksquare$

Whether 0 or 1 is an isolated cut point is also undecidable. However, this was open for sometime.

Consider the equivalence problem for probabilistic automata

Input : $A, B$

Equivalence: Is $[\![\mathcal{A}]\!] = [\![B]\!]$ ?

This problem is ~~[illegible]~~ decidable. Indeed, the automaton $[\![\mathcal{A}]\!] - [\![B]\!]$, which is no longer probabilistic, can be embedded in $\mathbb{R}$. Since $\mathbb{R}$ is a field, we may use techniques from LECTURE-4 to check support non-emptiness.