B1/BVR Probability HA1 04-01-2017

In soils of sand, the more you delve the more rush it springs
So too in learning, the deeper you go the more bounty it brings.
Tirukkural

1. A poker hand means a set of five cards selected at random from usual deck of playing cards.

   (a) Find the probability that it is a Royal Flush - means that it consists of ten, jack, queen, king, ace of one suit. **(b)** Find the probability that it is four of a kind - means that there are four cards of equal face value. **(c)** Find the probability that it is a full house - means that it consists of one pair and one triple of cards with equal face values. **(d)** Find the probability that it is a straight - means that it consists of five cards in a sequence regardless of suit. **(e)** Find the probability that it consists of three cards of equal face value and two other cards but not a full house. **(f)** Find the probability that it consists of two distinct pairs and another card but does not fall into previous categories. **(g)** Find the probability that it consists of a pair and three other cards but does not fall into previous categories.

2. In how many ways can eight rooks be placed on a chess board so that none can take another and none is on the white diagonal.

3. A number X is chosen at random from the set $\{0, 1, 2, \cdots, 10^n - 1\}$. Find the probability that X is a k-digit number. A number a is called a k-digit number if it is of the form $a = \sum_0^{k-1} a_i \, 10^i$ where a_i are integers from 0 to 9 and a_{k-1} is not zero.

4. One mapping is selected at random from the set of all mappings of $\{1, 2, \cdots, n\}$ to itself. What is the probability that the selected map transforms each of the n elements into 1?

   Let $1 \leq i, k \leq n$ be fixed. What is the probability that element i has exactly k pre-images?

   What is the probability that the element i is mapped to k? What is the probability that the elements i_1, i_2 and i_3 (assume distinct) are transformed to j_1, j_2 and j_3 respectively? (These i, j are given).

5. One permutation is selected at random from the set of all permutations of $\{1, 2, \cdots, n\}$. What is the probability that the identity permutation is chosen?

   What is the probability that the selected permutation transforms i_1, i_2 $\cdots$,i_k to j_1, j_2, $\cdots$,j_k respectively?

   What is the probability that the permutation keeps i fixed?

   What is the probability that the elements 1, 2, and 3 form a cycle in that order?(in some order?)

   What is the probability that all the elements form a cycle?

6. Tournament on n vertices is a directed graph, having exactly one arrow between each pair of vertices. A tournament is said to have k-leader property if for every set of k players there is one who beats them all. Consider a random tournament with n players. Fix k players.

   Show that the event "no player beats all these k", has probability $(1 - 2^{-k})^{n-k}$.

   Consequently, if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a k-leader tournment with n players.

   Show that for all large n this inequality holds and hence k-leader tournaments are possible.

   If $f(k)$ is the least such n, show that $f(1) = 3$ and $f(2) = 7$.

7. Select a point (x, y) at random from $\{1, 2, \cdots, n\}^2$. Let p_n be the chances that x and y are co-prime. Show $\lim p_n = (\sum \frac{1}{k^2})^{-1}$.

   [Thought easy, but may be difficult, shall explain in class.]

8. What is the probability that two throws with three dice each will show the same configuration, if the dice are distinguishable? what if dice are not distinguishable? (confusing?)

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Giving with respect to meet another's need ◇
And without ego, beats hollow the joys of receiving.
Don't give just so heavens counts that deed ◇
Giving's no giving when done to receive heavenly blessing. Tirukkural

9. Show $(AUB)^c = A^c \cap B^c$ and $ABC \subset (AB \cup BC \cup CA) \subset (A \cup B \cup C)$.

   For three events A, B, C find expressions for the following: **(a)** Only A occcurs. **(b)** both A and B occur but not C. **(c)** At least one occurs. **(d)** All three occur. **(e)** At least two occur. **(g)** One and no more occurs. **(h)** Two and no more occur. **(i)** Not more than two occur.

10. (Maxwell-Boltzman experiment; n boxes, r balls)

    Show: the probability that box 1 contains exactly k balls is given by $p_k = \binom{r}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{r-k}$.

    show that the most probable number of balls in box 1 is given by k where $\frac{r+1}{n} - 1 < k \leq \frac{r+1}{n}$.
    That is $p_0 < p_1 < \cdots < p_{k-1} \leq p_k > p_{k+1} > \cdots > p_r$.

    As $n \to \infty$ and $r \to \infty$ in such a way that $\frac{r}{n} \to \lambda$ show that for each m; $p_m \to e^{-\lambda}\frac{\lambda^m}{m!}$.

    Show that the probability of finding exactly m boxes empty equals

    $$p_m(r,n) = \frac{1}{n^r}\binom{n}{m}\sum_{\nu=0}^{n-m}(-1)^\nu\binom{n-m}{\nu}(n-m-\nu)^r.$$

11. (Bose-Einstein expt; n boxes, r balls)

    Show the probability that box 1 contains exactly k balls is given by $q_k = \binom{n+r-k-2}{r-k} / \binom{n+r-1}{r}$

    Let $n > 2$. Show that zero is the most probable number of balls in box 1. That is, $q_0 > q_1 > q_2 > \cdots$.

    As $n \to \infty$ and $r \to \infty$ in such a way that $\frac{r}{n} \to \lambda$ show, for each m; $q_m \to \frac{\lambda^m}{(1+\lambda)^{m+1}}$.

    Show the probability that 'exactly m boxes are empty' is given by $q_m(r,n) = \binom{n}{m}\binom{r-1}{n-m-1} / \binom{n+r-1}{r}$.

12. consider arrangement of r_1 alphas and r_2 betas in a row. A run is maximal consecutive sequence of a letter. For example $\alpha\alpha\beta\alpha\alpha\beta\beta\beta\beta$

has four runs: two alpha runs each of length 2 and two runs of beta of lengths 1 and 4.

Assume all outcomes are equally likely. The chances $p_{2\nu}$ of exactly 2ν runs and $p_{2\nu+1}$ of exactly $2\nu+1$ runs are given by

$$p_{2\nu} = \frac{2\binom{r_1-1}{\nu-1}\binom{r_2-1}{\nu-1}}{\binom{r_1+r_2}{r_1}}; \quad p_{2\nu+1} = \frac{\binom{r_1-1}{\nu}\binom{r_2-1}{\nu-1}+\binom{r_1-1}{\nu-1}\binom{r_2-1}{\nu}}{\binom{r_1+r_2}{r_1}}$$

Most probable number of runs is k where $\frac{2r_1r_2}{r_1+r_2} < k < \frac{2r_1r_2}{r_1+r_2} + 3$.

The probability of having k runs of alphas is $\pi_k = \frac{\binom{r_1-1}{k-1}\binom{r_2+1}{k}}{\binom{r_1+r_2}{r_1}}$.

13. In matching problem with N envelopes and letters, the probability of exactly m matches $(0 \le m \le n)$ is given by
$p_{[m]} = \frac{1}{m!}\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots\cdots \pm \frac{1}{(N-m-1)!} \mp \frac{1}{(N-m)!}\right]$.

14. Two similar decks of N distinct cards are matched against a similar target deck. Show: probability of exactly m double matches is
$u_m = \frac{1}{m!}\,\frac{1}{N!}\sum_{k=0}^{N-m}(-1)^k\,\frac{(N-m-k)!}{k!}$.
Show that $u_0 \to 1$ as $N \to \infty$. Show that $u_m \to 0$ as $N \to \infty$; for each $m \ge 1$.

15. Find the probability that in five tossings a coin falls heads at least three times in succession. Find the probability for a head run of at least length five in ten tossings.

16. Die A has four red faces and two white faces. Die B has two red and four white faces. A fair coin is flipped once; if Heads we use die A alone; if Tails we use die B alone.
Show probability of red at any throw is 1/2.
If first two throws resulted in Red, find the chances of red at third throw? If the first n throws resulted in Red, find the chances that die A is being used?

The earliest recorded statements of combinatorial rules appear in India. Sushruta of 600 BC stated: by combining the six tastes of swadu/amla; lavana/katu; usna/kashaya — sweet/sour, salty/bitter, pungent/astringent — we can make 15 double tastes, 20 triple tastes etc. Sixth Century Varahamihira did with larger numbers. Ninth Century Mahavira gave explicit general formula for the number of combinations.

Victor Katz

17. Picking card from usual deck with replacement: How many drawings are needed so that chances of obtaining at least one ace is at least $1/2$.

18. A test, developed to detect cancer, gives correct diagnosis for 98 percent of cancer patients. It gives wrong diagnosis for 4 percent of non-cancer patients. In a population 0.3 percent have cancer. If I select a person at random from the population and if the test said 'yes', what are the chances that the person has cancer?

19. Matching N envelopes and letters: What are the chances that there is match at place i given that there is no match at position j.

20. Have two urns each having 20 balls numbered $1, 2, \cdots, 20$. A set of six balls are drawn from each urn. What is the probability that exactly k numbers are common for both samples $(0 \leq k \leq 6)$?

    What if the six balls are drawn without replacement from each urn.

21. 20 balls are put in 20 boxes (Maxwell-Boltzman). Given that box 1 is empty, what are the chances that only one box is empty? Given that only one box is empty, what are the chances that box 1 is empty?

22. Polya scheme of adding c balls: Let $1 \leq m < n$ be integers and A, B each stand for any of red/green. Show (time symmetry)
    $P(A \text{ at draw } m | B \text{ at draw } n) = P(A \text{ at draw } n | B \text{ at draw } m)$.

23. Let $p_k(n)$ be chances of exactly k green balls in the first n draws in Polya scheme. Show [For ease in writing, interpret $p_{-1}(n) = 0$]

$$p_k(n+1) = p_k(n)\frac{r + (n-k)c}{g + r + nc} + p_{k-1}(n)\frac{g + (k-1)c}{g + r + nc}.$$

$$p_k(n) = \binom{-g/c}{k}\binom{-r/c}{n-k} \Big/ \binom{-(r+g)/c}{n} \qquad 0 \leq k \leq n.$$

24. A die is rolled twice. Let A : first throw is odd; B : second throw is odd; C : sum of the two throws is odd. Show that any two of these events are independent. Are A, B, C independent?

25. Let $G = (V, E)$ be a graph (given to you). Choose points of V at random, each with probability $1/2$. Let T be the random set so obtained. Call an edge 'crossing edge' if one end is in T and the other in $V - T$. Since T is random, the number X of crossing edges is a random variable. Show that $E(X) = e/2$ where e is the number of edges, that is, cardinality of E.

    Deduce that a graph with n vertices and e edges must have a bipartite subgraph with at least $e/2$ edges.

    Here a subgraph of (V, E) means (V^*, E^*) where $V^* \subset V$; $E^* \subset E$. A graph (V, E) is bipartite if $V = V_1 \cup V_2$ with (i) $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$; (ii) $V_1 \cap V_2 = \emptyset$ and (iii) every edge has one vertex in V_1 and other in V_2.

26. Consider a Erdos-Renyi random graph, where each edge is chosen, independent of others, with probability p. Say that a pair of vertices (unordered) is good, if either they are joined OR they are joined to a common vertex. Let X be the number of bad (not good) pairs. Since the graph is random, X is a random variable. Show that

$$E(X) \leq \binom{n}{2}(1-p^2)^{n-2}.$$

    Since we want to change n, since everything above depends on n; let us denote the above random variable by X_n instead of X.

    Show that $P(X_n \geq 1) \longrightarrow 0$ and thus $P(X_n = 0) \to 1$. This is expressed by saying: almost every graph has diameter 2.

    Diameter of a graph is the largest possible distance between pairs of vertices and distance between a pair of vertices is the number of edges in the shortest path joining them. If no such path exists between a pair of vertices then the distance between them is ∞.

27. Show that there is a tournament on n players which has $n!2^{-(n-1)}$ Hamiltonian paths. A Hamiltonian path is an ordering of the vertices $v_1, v_2, \cdots, v_n$ such that for each $i < n$, v_{i+1} beats v_i.

28. Let $v_1, \cdots, v_n \in R^n$ with $\|v_i\| = 1$ for all i. Show that there exist numbers ϵ_i $(1 \leq i \leq n)$ each ± 1 such that $\|\sum \epsilon_i v_i\| \leq \sqrt{n}$.

    Show that there exist numbers η_i $(1 \leq i \leq n)$ each ± 1 such that $\|\sum \eta_i v_i\| \geq \sqrt{n}$.

29. Consider placing 3 balls in 3 boxes (Maxwell-Boltzman). Let N be the number of occupied boxes. Let X_i be the number of balls in box i for $i = 1, 2, 3$.

    Calculate joint distribution of (N, X_3); of (X_1, X_2, X_3).

30. Let X, Y, Z be independent each $G(p)$. Calculate

$$P(X = Y); \quad P(X \geq 2Y); \quad P(X + Y \leq Z.)$$

    Let $U = \max\{X, Y\}$ and $V = X - Y$. Show that U, V are independent.

♣ ♢ ♡ ♠

Of the many slips on the slope of life's slippery slips ♢
The worst is the careless word that passes through your lips.

Tirukkural

31. Two dice are thrown. let X be the score on the first die and Y be the larger of the two scores. Find joint distribution of X, Y; their means, variances and covariance.

32. In five tosses of a fair coin, let X, Y, Z be, respectively, number of Heads; number of Head runs; length of largest Head run. Calculate their joint distribution. Make bivariate tables for each pair. calculate marginals. Find means variances and covariances.
(Yah, no Royal road, must plough through outcomes!)

33. Let $X_1, X_2, \cdots, X_k$ be iid uniform over $\{1, 2, \cdots, 100\}$. Let $U = \min\{X_i\}$ and $V = \max\{X_i\}$. Calculate $P(U \geq r; V \leq s)$ deduce formula for $P(U = r, V = s)$.

34. In a sequence of Bernoulli trials, let X be the length of the run (either success or failure) started by the first trial. Find its distribution, mean, and variance.

    Let Y be the length of the second run. Find its distribution, mean and variance. Find joint distribution of X, Y.

35. for a group of n people find the expected number of days of the year which are birthdays of exactly k people.

36. In system theory; a system is called 'k out of n' if it has n independently working units and the entire system works iff at least k of the units work. Here $1 \leq k \leq n$. suppose we have such a system and each unit has the same probability p of working. Using binomial probabilities, find the probability that the system is working.

37. We place n bar magnets end to end in independent random orientation. Like ends attract and unlike ends repel making the system into certain random number of blocks. Find expectation and variance of the number of blocks.

38. In rolling six true dice, what is the probability of obtaining at least one ace? probability of exactly one ace? probability of exactly two aces.?

39. How long should a sequence of random digits $(0, 1, 2 \cdots, 9)$ be, in order for the chances of digit 7 appearing to be at least $9/10$?

40. Suppose that $X_1, \cdots, X_{100}$ are independent random variables on a probability space (Ω, p). No calculations are needed below.

    (A) Take any three real valued functions f, g, h as follows: f defined on R^{30}; g defined on R^{50} and h on R^{20}. Define

    $U(\omega) = f(X_1(\omega), \cdots, X_{30}(\omega)) = fo(X_1, \cdots, X_{30})$.

    $V(\omega) = f(X_{31}(\omega), \cdots, X_{80}(\omega)) = fo(X_{31}, \cdots, X_{80})$.

    $W(\omega) = f(X_{81}(\omega), \cdots, X_{100}(\omega)) = fo(X_{81}, \cdots, X_{100})$.

    Show that U, V, W are independent. generalize.

    (B) Let π be a permutation of $\{1, 2, \cdots, 100\}$. Put $Y_i = X_{\pi(i)}$ for $1 \leq i \leq 100$. Show $\{Y_i\}$ are independent.

    (C) see what you get combining (A) and (B).

41. There is a population of N elements. We sample with replacement. Due to 'replacement' a sample of size 30, in general, contains less than 30 distinct elements. As the sample size increases newer and newer elements appear. Let S_r be the sample size necessary to get r distinct elements. Express S_r as sum of waiting times and show the following.

$$E(S_r) = N \left[\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{N-r+1} \right].$$

$$\text{variance}(S_r) = N \left[\frac{1}{(N-1)^2} + \frac{2}{(N-2)^2} + \cdots + \frac{r-1}{(N-r+1)^2} \right]$$

In particular

$$\text{variance}(S_N) = N \left[\frac{1}{(N-1)^2} + \frac{2}{(N-2)^2} + \cdots + \frac{N-1}{1^2} \right]$$

Show that

$$\lim_N \text{ Variance} \left(\frac{S_N}{N} \right) = \sum_1^\infty \frac{1}{n^2} = \zeta(2).$$

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Watch your thoughts; they become words.
Watch your words; they become actions.
Watch your actions; they become habits.
Watch your habits; they become character.
Watch your character; for, it becomes your destiny. Upanishads

42. Let $X \sim P(\lambda)$ and $Y \sim P(\mu)$ independent. show $X+Y \sim P(\lambda+\mu)$.

    Show that the conditional distribution of X given $X+Y=n$ is $B(n, \lambda/(\lambda+\mu))$.

    Calculate conditional expectation of X given $X+Y$, denote it Z. calculate $E(Z)$ and verify $E(E[X|X+Y]) = E(X)$.

43. Let $X \sim B(n,p)$ and $Y \sim B(m,p)$ independent. Show $X+Y \sim B(m+n,p)$.

    Show that the conditional distribution of X given $X+Y$ is Hypergeometric. Verify, by calculating both sides, the equality $E(E[X|X+Y]) = E(X)$.

44. Let $X \sim G(p)$ and $Y \sim G(p)$ independent. Show $X+Y \sim NB(2,p)$.

    Show that the conditional distribution of X given $X+Y$ is 'uniform'.

45. let S_n be the number of matches with n cards (or letters-envelopes). We saw $E(S_n) = 1$ no matter what n is. Show $Var(S_n) = 1$.

46. Let a population contain N items of which N_1 good and N_2 defective. Take a sample of size n without replacement and X_n = number of defectives in the sample. Show

$$E(X_n) = n\frac{N_2}{N}; \quad Var(X_n) = n\frac{N_1 N_2}{N^2}\frac{N-n}{N-1}$$

    What happens to the above values as $N, N_2 \to \infty$ in such a way that $N_2/N \to p$ $(0 < p < 1)$ (n fixed).

    Consider the same population but now sample of size n with replacement. Then what are the corresponding mean and variance of X_n = number of defective items in the sample.

47. An urn contains 35 black and 65 white balls and 10 balls are drawn. (a set of 10 balls or 10 balls without replacement?). Let X be the number of white balls drawn. Find $E(X)$ and $var(X)$.

48. In a company 5 percent of the produce is defective. For a shipment of 10,000 items the manager promises: 'if more than a defectives are found then I refund the money'. How should the manager choose a so that he need not have to refund more than one percent of time.

49. Let X be a random variable taking values $\{0, 1, 2, 3, \cdots\}$. Show

$$E(X) = \sum_1^\infty P(X \geq n) = \sum_0^\infty P(X > n)$$

50. Have a die with chance of heads p_i for face i; for $\leq i \leq 6$. I roll it 40 times. Let X be the number of faces that do not turn up at all. Calculate $E(X)$ and $var(X)$.

51. A random variable X takes values $\{0, 1, 2, \cdots\}$. The probability generating function (pgf)of X is defined by

$$\varphi_X(t) = \sum_0^\infty P(X = k)t^k; \qquad -1 \leq t \leq 1$$

Show that the function is well defined for those values of t above. Use theorems of your analysis course to show that it is a continuous function in $[-1, 1]$; it is a differentiable function in $(-1, 1)$; it is twice differentiable etc. Give a formula for $\varphi^{(n)}(t)$.

Using the numbers $\{P(X = k) : k \geq 0\}$ we made the function φ. if someone gives you a pgf φ, can you make out what $P(X = k)$ is?

Calculate the pgf of $B(n, p)$; $P(\lambda)$ and $G(p)$ variables.

If X and Y are independent random variables taking non-negative integer values, show that $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$.

52. You know that $cov(X, Y) = 0$ does not imply independence. However, if X and Y each take only two values then show that $Cov(X, Y) = 0$ does imply they are independent.

53. In Polya urn scheme; starting with g green and r red and c added; let X_n be one or zero according as n-th draw is red or green. Show $\rho(X_n, X_m) = c/(g + r + c)$.

54. Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with variance σ^2. Let $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$, their mean. Show that

$$E\left[\frac{1}{n-1}\sum_1^n (X_i - \overline{X})^2\right] = \sigma^2.$$

55. Let $X_1, \cdots, X_n$ be i.i.d. random variables assuming strictly positive values. Assume that $E(X_1)$ and $E(\frac{1}{X_1})$ are both finite. Let $S_n = X_1 + \cdots + X_n$. Show that $E(\frac{X_i}{S_n}) = \frac{1}{n}$ for $1 \leq i \leq n$. Show that for $m \leq n$; $E(\frac{S_m}{S_n}) = \frac{m}{n}$ and $E(\frac{S_n}{S_m}) = 1 + a(n-m)E(\frac{1}{S_m})$ where $a = E(X_1)$.

56. Here is Random Walk. I toss a coin whose chance of heads in a single toss is p, where $0 < p < 1$. Heads up, I move one step forward; tails up I move one step backward (on the space described below). Write down the transition probabilities in the following cases. In each of these cases calculate the distribution of X_2 after choosing some initial starting point X_0. Do it for enough initial points to get an idea.
(a) The state space is all integers. This is RW or *unrestricted* RW.
(b) The state space is $\{0, 1, 2 \cdots\cdots\}$. If you are at 0 then move to 1. This is RW *reflected* at zero.
(c) The state space is $\{0, 1, 2 \cdots, 99\}$. If at 0 or at 99, do not move; stay there forever. This is RW with two *absorbing barriers*.
(d) State space is as in (c). If at 0, move to 1. If at 99, move to 98. This is RW with two *reflecting barriers*.
(e) State space is as in (c). Fix two numbers $0 < r_i < 1$ for $i = 0$ and $i = 99$. If at 0 stay there with probability r_0 and go to 1 with probability $1 - r_0$. Similarly do at 99 using r_{99}. This is RW with two *elastic* barriers.
(f) State space is as in (c). If at 0, move to 1. If at 99, stay there. This is RW with one absorbing barrier and one reflecting barrier.
(g) State space is as in (c). Treat 0 as state next to 99 and 99 as state preceding 0. Equivalently, think of the states arranged in a circular/cyclic order. There is no need to specify anything more for the motion to continue. This is *cyclic* RW.

Amazing variations! instead of coin, may want to throw a die and act. Instead of dimension one, may want to walk in higher dimensions; or may want to walk on a graph, from one vertex to another; or on groups from one point to another $\cdots\cdots$. Why not?

May all become Happy ⋄ May all be free from Illness;
May all see what is Auspicious ⋄ May no one Suffer. Upanishads

57. Describe communicating classes in the following chains whose transition matrices are given below.

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/2 & 0 & 0 & 1/4 \\ 0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/16 & 1/16 & 1/4 & 1/4 & 1/8 & 1/4 \end{pmatrix}$$

Incidentally, as far as discussing communication is concerned, is it necessary to have the full matrix before you? Instead of exact numbers in the above matrices I just put $*$ when there is a strictly positive quantity. Would this have been enough for you?

58. Consider a three state chain with states 0, 1, 2 and transition matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix}$ Calculate P^2, P^3 and P^4. Calculate P^n for every n.

59. Consider a chain with three states named AA, Aa and aa respectively with transition matrix $\begin{pmatrix} p & (1-p) & 0 \\ p/2 & 1/2 & (1-p)/2 \\ 0 & p & (1-p) \end{pmatrix}$. Here $0 < p < 1$. Calculate the two step transition matrix, three step matrix. What happens in the long run?

60. A store near my house stocks Lyril soap. He follows (2/5) inventory schedule. This means, if he has two or less soaps by the closing time today, he will get some more from his godown to start with a total of 5 tomorrow morning. If he has more than two soaps at closing time today, then he does not replenish. Let us assume that the demand for this soap on successive days are i.i.d random variables; taking value i with probability p_i for $0 \leq i \leq 6$.

    Show that X_n = closing stock on n-th day; Y_n = opening stock on n-th day are Markov chains. Find their transition matrices.

61. State space consists of the 64 squares of a chess board. There is only one piece on the board, say, a knight on one square. from a state, you select one of its possible moves at random and move there. Is this irreducible chain? What is the mean time taken by the knight to return to the starting position?

    Do the same thing with another chess piece of your choice.

62. State space consists of the 52! arrangements of the usual deck of playing cards. Start with any one arrangement (think of vertical stack). Here are some laws of motion. In each case explain communicating classes.

    Move bottom card to the top.

    Interchange bottom and top cards!

    Pick 7th card from top (counting top card as one) and 9th card from bottom (counting bottom card as one). Interchange them.

    Pick two cards at random and interchange them. First do with replacement. Next consider without replacement.

    Cut at random and interchange the stacks. That is, pick a number at random from among $\{1, 2, \cdots, 51\}$. The stack is now split into two parts: top $[1, k]$ and bottom $[k + 1, 52]$. New stack is

    new 1=old $(k + 1)$; new 2 = old $(k + 2)$; new $(52 - k)$ = old 52; and

    new $(52 - k + 1)$ = old 1, etc new 52 = old k.

63. Consider the sliding board: state space = $\{0, 1, 2 \cdots\}$. if at 0, go to i with probability p_i where $\sum_{i \geq 0} p_i = 1$. If at $i \geq 1$, then move to $i - 1$. Show that there is a stationary distribution iff $\sum ip_i < \infty$ and in that case there is a unique stationary distribution.

64. Periodic observations of a markov chain form again a markov chain. More precisely, Let $(X_n, n \geq 0)$ be a markov chain. Consider $Y_n = X_{7n}$. Then $(Y_n, n \geq 0)$ is markov chain. Calculate its transition matrix.

More generally, let $Y_n = X_{31+59n}$. Show that $(Y_n, n \geq 0)$ is a markov chain. What is its Transition matrix and initial distribution?

65. Suppose that (X_n) is a Markov chain with state space S and transition matrix P.

Define $Y_n = (X_n, X_{n+1})$. Show that the Y process is a Markov chain with state space $S^2 = \{(s_1, s_2) : p_{s_1 s_2} > 0\}$. Calculate its transition matrix.

More generally, define $Y_n = (X_n, X_{n+1}, \cdots X_{n+L-1})$. Show that the Y process is a Markov chain with state space $S^* =$ finite sequences $(s_1, s_2, \cdots s_L)$ of length L from S such that $p_{s_1 s_2} p_{s_2 s_3} \cdots p_{s_{L-1} s_L} > 0$. Calculate its transition matrix.

If the original chain is irreducible then show that this new chain is also irreducible.

Moreover if the original chain has stationary distribution π then show that this chain also has a stationary distribution and get a formula for this in terms of π.

This Y chain is called the snake chain of length L associated with the X chain.

66. Consider Birth and Death chain with state space $\{0, 1, 2 \cdots, N\}$ with reflecting boundaries. Thus, from 0 you move to 1 and from N you move to $N - 1$. From any other state i, you move to $i-1$, i, $i+1$ with probabilities q_i, r_i and p_i respectively. Show that the stationary distribution is given by

$$\pi_i = \pi(0) \frac{p_1 p_2 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}; \quad i \geq 1; \qquad \pi_0 = ?$$

67. (Gambler's Ruin) consider random walk with two absorbing barriers; state space is $\{0, 1, 2 \cdots N\}$ and the transition mechanism is: win a rupee with probability p or loose a rupee with probability $1 - p$.

Let $u_i =$ probability of getting absorbed at N starting from i. Show $u_i = \dfrac{1 - [(1-p)/p]^i}{1 - [(1-p)/p]^N}; \quad p \neq 1/2;$ and $= i/N; \quad p = 1/2.$
Let t_i be the expected time for absorption starting from i. Argue $t_i = 1 + p m_{i+1} + q m_{i-1}$ with $m_0 = m_N = 0$. Show

$$t_k = \frac{k}{q-p} - \frac{N}{q-p} \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad \text{for } p \neq 1/2$$

whereas $t_k = k(N - k)$ for $p = 1/2$.

Do not be afraid of a small beginning. great things come afterwards.

Be courageous. Do not try to lead your brethren, but serve them.

The brutal mania for leading has sunk many a great ships in the waters of life.

Take care especially of that, i.e. be unselfish even unto death, and work.

Vivekananda

68. show: If $A \subset B$ then $P(A) \leq P(B)$. Show $P(A^c) = 1 - P(A)$.

    One of the rules of probability is that for disjoint sequence of events (A_n) we have $P(\cup A_n) = \sum P(A_n)$.

    If $B_n \uparrow B$ then show $P(B_n) \uparrow P(B)$. Here $B_n \uparrow B$ means the following: $B_n \subset B_{n+1}$ for all n and $\cup B_n = B$.
    (Express B as disjoint union and think).

    If $B_n \downarrow B$ then show $P(B_n) \downarrow P(B)$. Here $B_n \downarrow B$ means the following: $B_n \supset B_{n+1}$ for all n and $\cap B_n = B$.

    Show inclusion-exclusion formula holds. that is for any (finitely many) events $A_1, A_2, \cdots, A_n$

$$P(\cup A_i) = S_1 \ - \ S_2 \ + \ S_3 \ + \ \cdots \ \pm \ S_n$$

    (First recall what are these S_i).

69. Let X be any random variable (discrete or continuous). Define $F(a) = P(X \leq a)$ for $a \in R$. This function is called the Distribution Function (DF) of X.

    (i) Show F is monotone non-decreasing; $0 \leq F(a) \leq 1$.

    $\lim_{a \to -\infty} F(a) = 0$ and $\lim_{a \to \infty} F(a) = 1$.

    (ii) Show that F is right continuous.

    (iii) Show that F has left limit at every $a \in R$. That is, given $a \in R$; there is a number α such that whenever you take a sequence of numbers $x_n \uparrow a$; $x_n < a$ for every n, then $F(x_n) \to \alpha$. This number α is denoted $F(a-)$.

    (iv) Show that for every a; $P\{X = a\} = F(a) - F(a-)$.

    Deduce that $P\{a\} = 0$ iff F is continuous at a, equivalently, F is left continuous at a.

70. Let X be Bernoulli(p). Calculate its DF and see it.

Let $X \sim B(2, p)$ Calculate its DF and see it.

Do the same for poisson(λ) variable; for Uniform$(0, 1)$ variable; for exp(1) variable.

Let X be $N(0, 1)$. Can you calculate its DF explicitly and see?

71. Here is a nice DF. Let us enumerate the set of all rationals: $\{r_n : n \geq 1\}$. Let X be a random variable which takes the value r_n with probability $1/2^n$ for $n = 1, 2, \cdots$. Let F be its DF. There is no closed formula for this.

But show that F is continuous at a point iff it is irrational: (i) by using earlier results (ii) directly verifying definition of continuity, not using earlier results.

Thus there is a function on R whose set of continuity points is precisely the set of irrationals. Do you think there is a function on R whose set of continuity points is precisely the set of rationals?

72. Let X be Uniform$(0, 1)$. Find the density of $Y = 1 - X$. Find the density of $Z = -\log X$.

73. Show that the following function is a DF.

$F(x) = 0$ for $x < -1$; $F(x) = 0.25$ for $-1 \leq x \leq 0$;

$F(x) = 0.25 + 0.5x$ for $0 \leq x \leq 1$; $F(x) = 0.75$ for $1 \leq x < 2$;

and finally $F(x) = 1$ for $x \geq 2$.

Show F is a DF. Suppose X is a random variable with this DF.

List all numbers a with $P(X = a) > 0$. Calculate

$P(-1 < X < 2)$; $P(-1 < X \leq 2)$; $P(-1 \leq X \leq 2)$; $P(-1 \leq X < 2)$.

Do you think X is a discrete random variable?

Do you think X has density?

74. I toss a coin whose chance of heads is p. If Heads, I pick a number at random from $(0, 1)$; if Tails, I pick a number at random from $(1/2, 3/2)$. Let X be the number selected. Calculate the DF of X.

75. Do you know Walli's product.

$$\frac{\pi}{2} = \lim_{m \to \infty} \frac{2^2}{1 \cdot 3} \frac{4^2}{3 \cdot 5} \frac{6^2}{5 \cdot 7} \cdots \frac{(2m-2)^2}{(2m-3)(2m-1)} \frac{(2m)^2}{(2m-1)(2m+1)}.$$

This is a story about four people named Everybody, Somebody, Anybody and Nobody. There was an important job to be done. Everybody was asked to do it. Everybody was sure that Somebody would do it. Anybody could have done it, but Nobody did it. Somebody got angry about it because it was Everybodys job. Everybody thought Anybody could do it but Nobody realized that Everybody would not do it. It ended up that Everybody blamed Somebody when Nobody did what Anybody could have done.

76. Let

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad -\infty < x < \infty.$$

Show that this is density function. This is called Standard Cauchy density.

If X is a random variable with the above density, does $E(X)$ exist?

Suppose I pick a number θ at random from $(-\pi/2,\ +\pi/2)$ and consider $X = \tan\theta$. Calculate density of X.

More generally, fix $\mu \in R$ and $\sigma > 0$. Show

$$f(x) = \frac{1}{\pi}\frac{\sigma}{(x-\mu)^2+\sigma^2}; \qquad -\infty < x < \infty$$

is a density.

Sketch these for $\mu = 0, \sigma = 1$; for $\mu = 1, \sigma = 5$; for $\mu = 1, \sigma = 1/2$.

77. Show

$$f(x) = \frac{1}{2}e^{-|x|}; \qquad -\infty < x < \infty$$

is a density function. Sketch this curve. This is called double exponential or Laplace density function.

78. Let X be unif$(0,1)$. Define Z to be zero or one according as $X \leq 1/2$ or $X > 1/2$. Find the distribution of Z.

Exhibit a function of X (as above) which is $B(3,p)$.

Exhibit a function of X which is Poisson(λ).

79. Let X be a random variable with density $f(x) = 2x$ for $0 < x < 1$ and $f(x) = 0$ for $x \notin (0,1)$. Give a function of X, which is Unif$(0,1)$.

Let $X \sim \exp(1)$. Show a function of X which is Unif$(0,1)$.

80. Calculate the means and variances of the following random variables:

Unif(a,b); exp(λ); gamma(a); beta(a,b); $N(0,1)$; Laplace.

81. Show that the following is density. Here $\mu \in R$ and $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)}; \quad -\infty < x < \infty$$

This is called $N(\mu, \sigma^2)$ variable.

Suppose $X \sim N(0,1)$ and $Z = \sigma(X + \mu)$. Calculate the density of Z.

82. Fix an integer $r > 1$. Show that every number $x \in (0,1)$ has an expansion

$$x = \frac{z_1}{r} + \frac{z_2}{r^2} + \frac{z_3}{r^3} + \cdots\cdots ;$$

where each $z_i \in \{0, 1, 2, \cdots, r-1\}$. This is called expansion of x with base or radix r.

How many numbers have two expansions? How many have three expansions?

83. There is a list of k books. Each day, one of these k possible books are requested – the i-th one with probability P_i. Assume $P_i > 0$ for each i. The list is revised as follows: The book selected today is moved to the top of the list, while the relative positions of all the other books remains unchanged. The state of the system is the ordered list of the books. Note that there are $k!$ states. How does the transition matrix look like? Discuss periodicity.

For any state $\pi = \langle \pi(1), \pi(2), \cdots, \pi(k) \rangle$; let $\sigma(\pi)$ denote its limiting probability. In order to be in this state, it is necessary that the last request was for $\pi(1)$; the last non-$\pi(1)$ request be for $\pi(2)$; the last non-$\{\pi(1), \pi(2)\}$ request be for $\pi(3)$ etc. So it appears intuitively that

$$\sigma(\pi) = P_{\pi(1)} \frac{P_{\pi(2)}}{1 - P_{\pi(1)}} \frac{P_{\pi(3)}}{1 - P_{\pi(1)} - P_{\pi(2)}} \cdots$$

Show that σ is indeed stationary distribution.

84. We can consider oriented random walk on 'discrete torus'. Let $a > 1, b > 1$ be integers. Let $S = \{m, n) : 0 \le m \le a-1; 0 \le n \le b-1\}$. ¿From (x,y) you move to $(x+1, y)$ or $(x, y+1)$ with equal probability. Here addition is modulo a for first coordinate and modulo b for the second.

Show that the chain is irreducible. Show that it is aperiodic iff g.c.d (a,b) =1.

Take Risks in Your Life.

If You Win, You Can Lead. If You Loose, You Can Guide. Vivekananda

85. Sketch each of the following DFs. Get their densities and sketch them. (I have not defined them on all of R, what would you do?)

    (a) $F(x) = x/5$ for $0 \leq x \leq 5$.

    (b) $F(x) = e^{3x}$ for $x \leq 0$.

    (c) $F(x) = (2/\pi)\sin^{-1}(\sqrt{x})$ for $0 \leq x \leq 1$.

    (d) $F(x) = (x^3/2) + (1/2)$ for $-1 \leq x \leq 1$.

86. The random variable X is uniform $(-\alpha, \alpha)$ where $\alpha > 0$. For each of the following determine α so that stated equality holds:

    (a) $P(X > 1) = 1/3$. (b) $P(X > 1) = 1/2$. (c) $P(X < 1/2) = 0.7$.

    (d) $P(X < 1/2) = 0.3$. (e) $P(|X| < 1) = P(|X| > 1)$.

    Answer the same question if X is uniform over $(0, \alpha)$.

87. A continuous random variable X has density

    $f(x) = 3x^2$ for $-1 \leq x \leq 0$.

    If b is a number in this interval find $P(X > b \| X < b/2)$.

88. I toss a coin with chance of heads p. if I get heads I select a point at random from $(0,\ 1)$; if I get tails I select a point from $(1,\ 2)$ at random. Let X be the selected point. find the DF of X. Find its density.

    Suppose I change the above experiment as follows: if heads up I select a point at random from $(0,\ 2)$ and if tails up, I select a point at random from $(1,\ 3)$. Answer the same questions as above.

89. Let X be the life length of an electron tube and suppose that it can be thought of as an exponential variable with parameter λ. Let Y be the integer part of X. Calculate the distribution of Y.

    Find the density functions of $\sqrt{X}$; X^2; $1 - e^{-\lambda X}$; $(\log X)/\lambda$.

90. Let X be the life length of an electronic device measured in hours. Suppose it has density $f(x) = c/x^n$ for $2000 \leq x \leq 10000$.

    Determine c. Sketch the DF.

    What is the probability that the device will fail before 5000 hours.

91. A point is chosen at random from a line of length L. What is the probability that the ratio of the shorter to the longer segment is less than 1/4?

92. Suppose that the life length of a radio tube can be modelled as continuous random variable X with density $f(x) = 100/x^2 \quad \text{for} \quad x > 100$.

    What is the (conditional) probability that a tube will last more than 200 hours if it is known that it is still functioning after 150 hours of service.

    What is the probability that if three such tubes are installed in a set, exactly one will have to be replaced after 150 hours of service.

    What is the maximum number of tubes that may be inserted into a set so that there is a probability of at least 0.25 that after 150 hours of service all of them are still functioning?

93. If the random variable K is uniformly distributed over $(0, 5)$, what is the probability that the roots of the following equation are real?

$$4x^2 + 4xK + K + 2 = 0$$

94. Test your understanding of the normal number theorem by proving the following: If you pick a number at random from $(0, 1)$ then in its binary expansion each of the digits occurs with frequency 1/2. In other words, the chances that the selected number does not satisfying this condition are zero. (Stare at the steps, no further writing!)

95. To get practice Carry out the details of the following. Let $r > 1$ be an integer. Then for a point selected at random from $(0, 1)$, in its r-expansion, each digit $\{0, 1, 2, \cdots, (r-1)\}$ appears with frequency $1/r$. (Stare at the steps, no further writing!):

96. Return to decimal expansion. Let abc be a pattern of decimal digits, that is, it is an ordered triple of digits a, b, c each from zero to 9.

    Suppose $x = 0.x_1x_2x_3 \cdots x_n \cdots$. We say that the pattern occurs at place n if $(x_{n-2}x_{n-1}x_n) = (abc)$. Let $p_n(x)$ be the proportion of times the pattern occurs till n. That is

    $p_n(x) = \#\{m \leq n : \text{pattern occurs at place } m\}/n$. You may take, just for clarity, $p_1(x) = p_2(x) = 0$. Let X be a number selected at random from $(0, 1)$. Show that $p_n(X) \to 1/1000$ 'surely'.

    (Stare at steps; redefine Z_i; Think of bounding $E[(\sum U_i)^4]$)

97. I have L boxes and lots of balls with me. I put the balls, one by one, in the boxes by picking a box at random independent of previous choices. Let X_n be the number of empty boxes after n placements. Thus $X_0 = L$. Show that (X_n) is a Markov chain. describe the state space and transition matrix. Find expected time for reaching state 0.

98. I have L black balls and L white balls and I have two urns. To start with, I put L balls in each urn. Every minute, one ball is drawn *at random from each urn and put into the other urn.* draws are independent at each stage. Let X_n be the number of black balls in urn 1 after n exchanges. Describe the state space and transition matrix. Show that the chain is irreducible and aperiodic. Show that the stationary distribution is given by

$$\pi_k = \binom{L}{k}^2 \Big/ \binom{2L}{L} \qquad k = 0, 1, \cdots, L.$$

99. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from front or back door. While leaving from the door he chooses a pair of running shoes at that door. Of course, he goes barefoot if there are no shoes at the departing door. On his return he is equally likely to enter through front or back door, where he then leaves the shoes he is wearing. If he owns a total of k pairs of shoes what proportion of time does he run barefooted?

100. A die is consecutively turned from one face to any of the four neighbouring faces with equal probability and independent of the preceding turns. Find $\lim_n p_{66}^{(n)}$.

101. (*Metropolis Chain*) Want to simulate a probability π on a finite set S; $\pi(s) > 0$ for all s.

First make a connected graph on S, let $d(v)$ be the degree of the vertex v in the graph. Here is a Markov Chain.

If you are at v, pick a neighbour w of v at random. Move there with probability $\min\{1, \pi_w d_v / \pi_v d_w\}$. With remaining probability stay at v itself.

Show that this gives aperiodic chain and simulates π. If you choose simple graphs, the simulation would be easier.

But we are convinced that if we are to play a meaningful role nationally, and in the community of nations, we must be second to none in the application of advanced technologies to the real problems of man and society. Vikram Sarabhai

102. A r.v. X has pgf φ. Find pgfs of $X+1$ and $2X$ in terms of φ

103. Find the gfs of following in terms of pgf of X.
 $\{P(X \leq n)\}$; $\{P(X < n)\}$; $\{P(X > n+1)\}$; $\{P(X = 2n)\}$.

104. Suppose that X and Y are independent random variables each taking values $\{1,2,3,4,5,6\}$. Suppose

$$P(X=j) = a_j; \qquad P(Y=j) = b_j; \qquad 1 \leq j \leq 6$$

 Show (using zeros of polynomials) that φ_{X+Y} can not equal

$$\frac{t^2 + t^3 + t^4 + \cdots + t^{12}}{11}$$

 You can never have two 'loaded dice' so that when you roll them, all possible scores are equally likely! score means sum of the two faces.

105. Consider random walk starting from zero, but moves are governed by a coin with chance of heads p. Let a_n be the probability that first visit to 1 occurs on day n. We take $a_0 = 0$; clearly $a_1 = p$. Show for $n > 1$,

$$a_n = q(a_1 a_{n-2} + a_2 a_{n-3} + \cdots + a_{n-2} a_1)$$

 Show that

$$\varphi(t) - pt = qt\varphi^2(s)$$

 Deduce

$$\varphi(t) = \frac{1 - \sqrt{1 - 4pqt^2}}{2qt}; \qquad \text{hence} \quad \varphi(1) = \frac{1 - |p-q|}{2q}$$

 conclude the following:

 if $p \geq q$ then you are sure to visit 1.

 if $p < q$ then then the chance of ever visiting one equals p/q.

106. In a sequence of Bernoulli trials, define $X = n$ if the combination SF occurs for the first time at trials $n-1$ and n. Thus $X \geq 2$. Show

$$\varphi(t) = \frac{pqt^2}{(1-pt)(1-qt)}; \quad E(X) = 1/pq; \quad var(X) = \frac{1-3pq}{p^2q^2}$$

107. Suppose I have a random variable with moments μ_k, that is, $\mu_k = E(X^k)$ for $k \geq 0$, where, of course, $\mu_0 = 1$. Show that the following matrix is non-negative definite.

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{k+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k} \end{pmatrix}.$$

108. Suppose that X is a nonnegative random variable. Fix a number $\delta > 0$. Define the following random variable X_δ:

X_δ takes the value $k\delta$ iff $k\delta \leq X < (k+1)\delta$; for $k = 0, 1, 2 \cdots$.

This is called " discretization" in mathematics, "grouping" in statistics and "quantization" in physics.

Assume that X has density f which is bounded continuous on a bounded interval and zero outside the interval. Show rigorously that $E(X_\delta)$ converges to $\int x\, f(x)\, dx$ as $\delta \to 0$. What if the density does not have bounded support?

109. Suppose that a random variable X has density f. Median of X is any number c such that

$$P(X \leq c) = \frac{1}{2} = P(X \geq c); \quad (i.e.) \quad \int_{-\infty}^{c} f(x)dx = \frac{1}{2}$$

(Can give one definition to cover both discrete and density cases, but do not bother). Assume that X has a unique median m.

For any real number b, show (assume needed integrals exist)

$$E(|X - b|) \quad = \quad E(|X - m|) + 2\int_m^b (b-x)\, f(x)\, dx$$

For what b is $E(|X - b|)$ minimum. What if median is not unique?

affection towards equals; appreciation towards superiors
sympathy towards the deficient; ignoring those hostile.
is the way to peace and happiness. Patanjali

110. (a) Have a coin whose chance of heads is p $(0 < p < 1)$. I toss it n times. Let X be the number of heads and Y be the number of tails. Are they independent?

(b) Suppose N is Poisson with parameter λ. I toss the coin N times. This means the following. I do a two stage experiment. First I pick a number from among $\{0, 1, 2, 3, \cdots\}$ so that, for each k, probability of picking k equals $e^{-\lambda}\lambda^k/k!$. Second stage consists of tossing the coin as many times as the number so selected.

Let X be the number of heads and Y be the number of tails obtained in this experiment. Show that X and Y are independent and $X \sim P(\lambda p)$ and $Y \sim P(\lambda q)$. is it not funny?

(c) Instead of coin, suppose I do this experiment with a die at the second stage. Assume that chance of face i is p_i for $1 \le i \le 6$. In this experiment, let X_i be the number of times face i appears.Show $X_1, X_2, \cdots, X_6$ are independent. Find their distributions.

111. I have the set $S = \{1, 2, \cdots, 100\}$. I want to pick a subset of S at random. Remember there are 2^{100} subsets.

Someone told me the following. You just toss a fair coin 100 times. Let A consist of all i such that i-th toss resulted in heads. Then A is a 'random set'. Show this.

112. As earlier $S = \{1, 2, \cdots, 100\}$. Among all subsets of S with cardinality 50, I want to pick one at random. Remember there are $\binom{100}{50}$ such subsets.

Some one told me the following. Pick a permutation π of S at random. Then independently pick $1 \le i \le 100$ at random. Then the following set is such a set: $A = \{\pi(i), \pi(i+1), \cdots, \pi(i+49)\}$. Show this.

113. As earlier $S = \{1, 2, \cdots, 100\}$. I want to pick a permutation at random. Instead of writing 100! permutations and doing this, someone said I can

do the following. Do sampling without replacement (size 100) from S. The ordered list so obtained is a random permutation. Show this.

In this second experiment, you are selecting 100 times; but each time from a small manageable set.

114. I want to pick a number at random from $(0, 1)$. This is a huge task. I can only approximate. I fix an error bound, say, 0.0001. If X is the number picked then, instead of demanding that $P(X \in I)$ should equal length of I; we only demand that the difference between these two must be at most 0.0001 *whatever* be the interval.

    Someone told me the following. Fix a large number k. You toss a fair coin k times and think of the result as dyadic expansion of a number. This will do. Show this. How large should I fix k. What if the error bound is 2^{-30}.

115. A subset A of integers is said to be sum-free, if you *can not* find $a_1, a_2, a_3 \in A$ (I did not say distinct) such that $a_1 + a_2 = a_3$. Thus if $0 \in A$, then A can not be sum free.

    Given a set B of non-zero integers, we can find a sum-free subset $A \subset B$ having at least one-third of the number of elements of B. Here is how to do it.

    Say, $B = \{b_1, \ldots, b_n\}$. Pick a prime $p > 2 \max\limits_{1 \le i \le n} |b_i|$ which is of the form $p = 3k + 2$. Show that such a p exists.

    Show $C = \{k+1, k+2, \ldots, 2k+1\}$ is sum-free in $Z_p = \{0, 1, \ldots, p-1\}$ where addition is done modulo p.

    Pick x at random from the set of non-zero elements of Z_p and set $d_i = x \cdot b_i \ mod\ (p)$. Show that, for fixed i, as x ranges over the non-zero elements of Z_p, so does d_i. Conclude that $P(d_i \in C) \geq 1/3$.

    Show that the expected number of b_i such that $d_i \in C$ is at least $n/3$.

    Conclude that there is x, $1 \le x \le p - 1$ and an $A \subset B$ with $|A| \ge n/3$ such that $x \cdot a \in C$ for all $a \in A$. Here $x \cdot a$ is taken mod (p).

    Show that A is sum-free. (See the power of probability!)

The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. S Chandrasekhar

116. Let X and Y be independent $uniform(0,1)$ random variables. For each real number a calculate $F(a) = P(X+Y \leq a)$.

   [Hint: Do for $a \leq 0$; $0 \leq a \leq 1$; $1 \leq a \leq 2$; $a \geq 2$.]

   Show that $X+Y$ has density f. Sketch its graph. Do you feel it can be called triangular density?

117. Consider a finite state Markov chain. Fix any state i. Fix it. Show that there is a state j such that $\sum\limits_n p_{ij}^{(n)} = \infty$

   conclude $\lim\limits_{t\uparrow 1} P_{ij}(t) = \infty$. Deduce $\lim\limits_{t\uparrow 1} P_{jj}(t) = \infty$

   Show: A finite state Markov chain has at least one recurrent state. If irreducible, all states are recurrent. Is this true for infinite state chains?

118. Consider an irreducible Markov chain on a finite set S with transition matrix P.

   Show: there is integer R such that: $\forall(i,j \in S)\ \exists(m \leq R)\ P_{ij}^{(m)} > 0$.
   Show: there is an $\epsilon > 0$ such that: $\forall(i,j \in S)\ \exists(m \leq R)\ P_{ij}^{(m)} > \epsilon$.

   Now fix a states $t \in S$. Let T be the first visit to t after time zero. That is for any sample point ω, $T(\omega)$ is the least $m \geq 1$ such that $X_m = t$. In other words, $X_m = t$ and $X_n \neq t$ for $1 \leq n < m$. This is well defined with probability one (see previous Ex). Plan: to show $E_s(T) < \infty$ for any s. so Fix s.

   Show $P_s(T > R) \leq (1-\epsilon)$.
   Show, for $m \geq 1$; $P_s(T > mR) \leq (1-\epsilon)P_s(T > (m-1)R)$.
   Deduce $P_s(T > mR) \leq (1-\epsilon)^m$.

   Show that $\sum\limits_{m\geq 1} P_s(T = m) = 1$. Show $E_s(T) < \infty$.

119. Here is, what is called, *birth and death chain*: $S = \{0, 1, 2 \cdots, R\}$. For each i there are given non-negative numbers p_i, q_i, r_i, adding to one for each i. Further, $q_0 = 0 = p_R$. all other p and q are strictly positive.

   here is the motion: If at i, move one step forward with probability p_i; one step back with probability q_i; stay at i with probability r_i.

Put $a_0 = 1$ and for $k \geq 1$, $a_k = \prod\limits_{i=1}^{k} \frac{p_{i-1}}{q_i}$. Show that normalization of the vector $(a_i : 0 \leq i \leq R)$ is the invariant probability.

120. Consider a finite group G and a probability μ on G. Here is random walk driven by μ on G: if at g, select h according to μ and move to hg. That is, the transition matrix is $P(g, hg) = \mu\{h\}$. Equivalently $P(g, g^*) = \mu\{g^* g^{-1}\}$

Show that this chain is irreducible iff $\{x : \mu\{x\} > 0\}$ generates the group G. Show that uniform probability on G is invariant probability for this chain.

121. Let $P = \{p_i : i \in S\}$ and $Q = \{q_i : i \in S\}$ be two probabilities on a finite set S. remember for events A, $P(A) = \sum\limits_{i \in A} p_i$ and similarly $Q(A)$ is defined. The total variation distance between P, Q is defined by

$d(P, Q) = \sup\{|P(A) - Q(A)| : A \subset S\}$.

Show $d(P, Q) = \frac{1}{2} \sum_i |p_i - q_i|$. Show

$d(P, Q) = \frac{1}{2} \sup \left\{ \sum\limits_i f(i) p_i - \sum\limits_i f(i) q_i : f : S \to R; \quad (\forall\, i)\ |f(i)| \leq 1. \right\}$

Show $d(P, Q) = \inf\{P(X \neq Y)\}$ where the infimum is taken over all pairs of random variables (X, Y) such that $X \sim P$ and $Y \sim Q$. Actually this inf is minimum.

A pair of random variables (X, Y) defined on one probability space such that $X \sim P$ and $Y \sim Q$ is called a *coupling* of the two probabilities P and Q.

122. $X \sim Poisson(n)$. If n is very large explain why we can approximate the probability $P(n + a\sqrt{n} < X < n + b\sqrt{n})$ by the number $\int\limits_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Here $a < b$.

Show that as $n \to \infty$,

$$\sum_{k \geq n} \frac{n^k}{k!} e^{-n} \to \frac{1}{2}.$$

Our greatest weakness lies in giving up.

The most certain way to succeed is always to try just one more time.

Thomas Edison

123. Given $-\infty < a < b < \infty$; Given a real continuous function f on $[a,b]$; Given $\epsilon > 0$; exhibit a polynomial P such that
$\sup\{|f(x) - P(x)| : a \leq x \leq b\} < \epsilon$.

Deduce the following: Given a continuous $f : R \to R$; there is a sequence of polynomials $\{P_n\}$ such that
$P_n \to f$ uniformly *over every bounded interval.*

124. Alphabet: $1, 2, 3, 4$.

The code $\{0; 010; 01; 10\}$ is not uniquely decipherable code. Argue

The code $\{10; 00; 11; 110\}$ is uniquely decipherable. Show.

The code $\{0; 10; 110; 111\}$ is prefix free code; that is, no code is the beginning segment of another code. Such codes are also called instantaneous codes. Show such codes are uniquely decipherable.

125. Alphabet: $\{1, 2, 3, 4\}$ with probabilities $P = \{1/2; 1/4; 1/8; 1/8\}$.

Consider the code $\{0; 10; 110; 111\}$. Calculate entropy $H(P)$ and expected code length.

126. Alphabet: $\{1; 2; 3\}$ with uniform probability P and code $\{0; 10; 11\}$. Calculate $H(P)$ and expected code length.

127. Let $(X_i : i \geq 1)$ be a sequence of bounded random variables each having the same distribution. Assume that for each n the random variables $\{X_1, X_2, \cdots, X_n\}$ are independent. Let $E(X_1) = \mu$. Let $A_n = \sum X_i/n$.

Following the same argument as in the frequency of decimal digits, show that for each $\epsilon > 0$, we have $\sum P(|A_n - \mu| > \epsilon) < \infty$.

Show that the average $A_n = \sum_1^n X_i/n$ converges to μ surely.

128. Consider a Markov chain, state space $\{0, 1 \cdots, K\}$. Assume $p_{00} = 1$. Such a state is called absorbing. Assume that for each $i \neq 0$; $i \rightsquigarrow 0$.

Show that 0 is recurrent, all other states are transient state.

Show that for each i; $P_i(X_n = 0 \quad \text{eventually}) = 1$.

In general show the following. Suppose i, j are states; $i \rightsquigarrow j$ but $j \not\rightsquigarrow i$ then i is a transient state. Also in a finite state chain, if a state i is transient then there must exist a state j such that $i \rightsquigarrow j$ but $j \not\rightsquigarrow j$.

129. Consider a chain with three states aa, Aa, AA and transition matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$. Find the probability of getting absorbed in each of the two absorbing states; starting from Aa. If the middle is row is $(p\ r\ q)$ then what would be the answer?

130. Suppose you have a chain with K states; accordingly a $K \times K$ transition matrix P. Suppose that this matrix has a real eigen basis. That is, K linearly independent vectors $(v_i : 1 \leq i \leq K)$ of R^K and numbers (λ_i) such that $Pv_i = \lambda_i v_i$. Then it is easy to calculate the n-step transition matrix P^n. Explain how.

    Show that eigen values of P have modulus at most one. Show $\lambda_1 = 1$ is an eigen value of P.

    It is possible for P to have non-real eigen values. Consider the chain with thirty states $\{0, 1, \cdots, 29\}$ and the transition mechanism is: from i you surely go to $i + 1$ (modulo 30). What are its eigen values?

    It is possible for P to have eigen values of modulus one, other than one.

    It is possible for P to have one as an eigen value of any given multiplicity. Give examples.

    If one is an eigen value of multiplicity 5, show that there are 5 communicating classes.