A TRIBUTE TO PROF. C.S. SESHADRI

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My relationship with Prof. C.S. Seshadri extends over five decades, first as my mentor, next as a collaborator, and then as a friend. Let me dwell upon these three phases, highlighting the important events that took place, during each of the phases.

I. Seshadri as my mentor: I joined TIFR in the Fall of 1968, the iconic year, which witnessed the following two important activities: the first one being the "celebrated" International Symposium on Algebraic Geometry in Jan 1968, and the second one being, the "famous" year-long course on Algebraic Geometry given by Seshadri. Unfortunately, I couldn't participate in either one of them for obvious reasons. After spending a couple of years, learning about the rudiments of the three basic courses Algebra, Analysis and Topology, during 1971, I formally started to work on my Ph.D,under the guidance of Seshadri. In the Fall of 1972, Seshadri asked me and Musili to read the paper " Schubert methods with an application to algebraic curves." by Kempf ([3]). In this paper, Kempf proves the Borel-Weil theorem for the Flag Variety $SL(n)/B$.in positive characteristics (let us recall that the celebrated Borel-Weil theorem is about the vanoshing of higher cohomologies of ample line bundles on the generalized Flag Variety $G/B, G$, a semi-simple algebraic group, and B, a Borel subgroup of G, in characteristic 0). Musili and I concentrated for a whole month on reading Kempf's paper and among other things, we arrived at the Weyl-group theoretic description of a class of smooth Schubert varieties in $SL(n)/B$, constructed by Kempf in that paper. We then reported to Seshadri about our discoveries of Kempf's paper. Then the three of us had serious discussion of Kepf's paper for about three months or so and we gave a proof of Borel-Weil theorem for the generalized Flag Variety $G/B, G$ being a group of classical type or type G_2.in all characteristics. This result "Cohomology of line bundles on G/B" was published in Ann. Sci. E.N.S, 87A, 1974, 87-138 ([9]). This was my first/proud publication. In my thesis, I extended the results of this paper to the class of Kempf varieties (the varieties considered by Kempf, in type A, and their generalizations to other types B,C,D). It's worth-mentionlng that the Kempf varieties again play a crucial part in my paper with Sandhya

"A characterization of smooth Schubert varieties in $SL(n)/B$" ([15]); they again show up in [5], authored by M. Kummini, V. Lakshmibai, P. Sastry, C.S. Seshadri. Kempf made use of a technical step in our paper [9], and gave a type-free proof of the Borel-Weil theorem, for $G/B, G$, a simple algebraic group and B a Borel sub group in all characteristics in 1976 ([4]), now known as "Kempf's vanishing theorem".

II. Seshadri as my collaborator: My collaboration with Seshadri runs over a little more than four decades-starting from mid-seventies through 2019. In the Fall of 1976, when Seshadri had just returned from a two-year visit to Harvard University, Seshadri was all excited about a recent work of De Concini-Procesi ([1]), where they present a characteristic-free approach to Classical Invariant Theory; their work essentially consists of construction of a characteristic-free basis for the rings of invariants, appearing in Weyl's "Classical Groups". It's more or less around this time that Seshadri had just finished his work on "Geometry of G/P"-I (cf. [24]), wherein he extends Hodge's results - giving a natural basis (over $\mathbb{C}$) for the homogeneous co-ordinate ring of the Grassmannian (and its Schubert varieties) for the Plücker embedding, in terms of "standard monomials" in the Plücker co-ordinates -, to G/P, G being a simple algebraic group, and P being a maxtmal, minuscule, parabolic subgroup of G, in **all** characteristics. We may describe Seshadri as the *inventor of modern Standard Monomial Theory* (abbreviated as SMT, henceforth). This work of Seshadri may be considered as the beginning of SMT! As soon as he got back to the Institute, he asked me to read the above-mentioned paper of De Concini-Procesi, and at the same time he explained to me about his work "Geometry of G/P-I". After spending countless hours on mathematical computations and discussions with Seshadri, we could figure out the relationship between the two papers (over a period of three months), and along the way, we also arrived at some important basic conjectures. Thus we have the birth of Geometry of G/P-II ([16]), which is to be considered as the *Gateway* to SMT.

The conjectures, arrived as in [16], describe a nice conjectural basis, in all characteristics, for all G/P's, where G is a simple algebraic group and P is a maximal parabolic subgroup of *classical type*; in particular, note that any maximal parabolic sub group of a classical group G, being of classical type, this would take care of the problem of developing a SMT, for all G/P's, G being classical and P any maximal sub group of G.

Together with Musili, we started our discussion on proving the basic conjectures. As a first step, we were able to prove the conjectures, for the case of G/P's, G being simple, and P being a maximal parabolic sub group of *quasi-minuscule type* ([10]). These are the parabolic sub groups that are considered by Kempf in [4]. The point here is that unlike the class of minuscule parabolic sub groups which fail to cover all simple algebraic groups, for instance, there aren't any minuscule parabolic sub groups if G is of type E_8, F_4, or G_2, the class of the quasi-minuscule parabolic sub groups is wider than the class of minuscule parabolic sub groups, in that every simple algebraic group admits at least one maximal parabolic sub group of quasi-minuscule type. At this time M. Demazure was visiting T.I.F.R., and gave lectures on his "celebrated" paper "Desingularization of Schubert varieties"([2]). In this paper, among other things, Demazure proves a character formula for the space of sections of ample line bundles on G/B as well as their restrictions to the Schubert sub varieties in G/B; this character formula is used in a crucial way in [10]. In addition, the normality of Schubert varieties, in all characteristics (a consequence of the results in [2]) has also been used in a crucial way in [10].

The three of us - Musili, Seshadri and I - continued on our investigations on the extension of the results for $G/P, P$ being a maximal parabolic sub group of classical type to any G/Q, G a simple algebraic group, and Q, an intersection of maximal parabolic sub groups of classical type - such a parabolic sub group is called *a parabolic subgroup of classical type*. After an intense discussion for four months, by the three of us, we did arrive at a set of conjectures for developing a SMT, for the case of G/Q, Q being a parabolic subgroup of classical type. Then after a six-months period of intense/serious discussion by the three of us, we had [11], in which we had given the complete proof of the conjectures of [16], and wherein we had stated the conjectures for the case of G/Q, and also had outlined a proof of the same. I have very good reminiscences of our discussion: We used to meet around 12 noon and go on until 7 pm, with a lunch break for an hour and then a couple of tea-breaks. It should be remarked that most of the times, the crucial idea will strike us while we were having our tea! At this time, Procesi was visiting Tata Institute - Jan, Feb, 1978. Upon his invitation, I was visiting Rome University, for the academic year 1978-79, as a visiting Professor. Procesi, De Concini and I had very many interesting mathematical discussion. The following two years, at the invitation of Kempf, I was a visiting Prof at Johns Hopkins. The following three years, I spent at University, Ann Arbor. In the summer of 1982, on my way to India, I had made a stop in Germany to attend a week-long conference on "Algebraic Groups" in Oberwolfach, organized by Springer and Tits. During the conference, Victor Kac pointed out one serious error in [2]; apparently, when Kac was running a seminar at MIT on [2], they found this error. As soon as I reached Bombay, right after the Oberwolfach conference, during my meeting with Seshadri, I mentioned to him about the serious error in [2]. We got very worried, since, as mentioned above, we had used the results of [2] in our papers [10, 11] in quite a non-trivial way. Once again, we got into serious mathematical discussion. Thankfully, we were able to fix our proof by taking a totally different approach than that found in [11]. Thus, we had the paper "Geometry of G/P-V".

We were still thinking about the extension of SMT to other exceptional groups. In the process, I arrived at a set of conjectures (see [20], for a statement of these conjectures). Our goal was to extend the SMT for the Exceptional groups, but to our surprise, the conjectures seem to include the Kac-Moody groups also. As a first step towards proving the conjectures, Seshadri and I could show that the conjectures hold for $\widehat{SL(2)}$.

Thanks to the ingenuity of Littelmann , who completed the SMT even for the Kac-Moody groups by proving the above-mentioned conjectures (cf.[22]). Littelmann's proof makes a clever use of the representation theory of Quantum groups at a root of unity, as developed by Lusztig in 1990's.

Now that the SMT is complete, Seshadri and I were looking at the problem of relating the cotangent bundle to G/B to some suitable Schubert variety in the affine Flag variety. To make this a little more precise: Lusztig (cf. [23])relates certain orbit closures arising from the type A cyclic quiver $\widehat{A_h}$ to certain affine Schubert varieties. On the other hand, in the case $h = 2$, Strickland (cf. [25]) relates such orbit closures to conormal varieties of determinantal varieties; furthermore, any determinantal variety can be canonically realized as an open subset of a Schubert variety in the Grassmannian (cf. [16]). Inspired by these results, we were interested in finding a relationship between affine Schubert varieties and conormal varieties to the Grassmannian. As a first step, I was able to show that the compactification of the cotangent bundle to the Grassmannian is canonically isomorphic to a Schubert variety in a two-step affine partial flag variety (cf. [8]). This result was extended in [14] to cominuscule Grassmannians.

Then in 2017, together with Rahul, we (Seshadri and I) were able to extend these results to $SL(n)/B$ (cf. [21]). This is my last publication together with Seshadri and his last publication also!

III. Seshadri as a friend: After spending nearly three decades at T.I.F.R., Seshadri moved to Chennai in the mid 1980's, and joined

IMSc, Chennai, as the head of the Mathematics Department; at more or less the same time, I moved to Boston to join the Math. Faculty at Northeastern University. At this point, it should be remarked that both of the research institutions, IMSc and T.I.F.R., accepted students only at the PhD level. After spending a brief time at IMSc, Chennai, Seshadri branched out and started an institution with the mission of training undergraduates, besides undertaking research. This new Institution was formed, in October 1989, with the help of Parthasarathy, who was then working at the petrochemical company SPIC in Chennai, and the new institution was called the SPIC Mathematical Institute.

After going through some initial financial difficulties, this institution eventually evolved into the Chennai Mathematical Institute, abreviated as CMI. CMI has evolved internationally, as one of the most well-recognised Indian institutions for mathematics. The under-graduates from CMI are accepted for a Ph. D. program in such top-notch Universities as Harvard, M.I.T., Brandeis, Northeastern, Princeton, U. Emory, U. Chicago, Caltec, UCLA, etc., in U.S., I.H.E.S., Ecole Normale, and U. Paris 7, etc., in France, Max-Plank Institute, U. Hamburg, etc., in Germany.

Ever since the birth of CMI, in 1989, I have been a regular visitor at CMI, visiting CMI, during. the winter months, of every year which enabled our collaboration to continue!. Having known Seshadri for 5 decades, I would describe him as a very genteel person, patient to the core who brought nothing but joy & pleasure to the people around him. Of all the good traits about him, I like his PATIENCE the most. I have never seen him lose his patience, not only with me, but also the with the people around him! I could say that just by observing him, I have learnt so many things about life, which have polished my nature/character unknowingly! I could easily say that my character of what I am being today, I owe it to my association with him for 50 years. Of all the things that I imbibe from my association with him, I would like to mention the quality of staying cool, calm & collected at all times! I have heard his friends/colleagues describe him as being "a very calm and collected person". Another trait in him that can not be missed by anyone who might have had just an acquaintance with him is his modesty; in spite of being the recipient of so many glorious awards, he remained so modest that this trait of his modesty only added to his personality! I would like to end this article by quoting Seshadrt's general PHILOSOPHY ON LIFE:

"NOTHING IS THE END OF THE WORLD"

References

[1] C. De Concini and C. Procesi, *A characteristic free approach to Invariant Theory*, Advances in Math., 21, 330-354.

[2] M. Demazure, *Desingularization des varittes de Schubert gineralisees*, Ann. Sci. Ecole Norm. Sup. I (1974), 53-88.

[3] G. Kempf. *S*chubert methods with an application to algebraic curves, 1971. Math. Centrum, Amsterdam, Afd. zuivere Wisk. ZW 6/71, (1971).

[4] G. Kempf, *Linear systems on homogeneous spaces*, Ann. of Math. (2) 103 (1976), 557-591.

[5] M. Kummini, V. Lakshmibai, P. Sastry, and C.S. Seshadri *F*ree resolutions of certain Schubert singularities in the Grassmannian, PJM, 2015, special volume dedicated to the memory of Prof. R. Steinberg.

[6] V. Lakshmibai, *Kempf varieties*, J. Ind. Math. Soc., Vol. 40 (1976), 299-349.

[7] V. Lakshmibai, *Standard monomial theory for* G_2 , J. Alg., Vol. 98 (1986), 281-318.

[8] V. Lakshmibai, *Cotangent bundle to the Grassmannian Variety*, Transformation Groups, Vol. 21, No. 2, 2016, pp. 519-530.

[9] V. Lakshmibai, C.Musili and C.S.Seshadri, *Cohomology of lme bundles on* G/B, Ann. Sci. Ecole Norm. Sup. I (1974), 89-138.

[10] V. Lakshmibai, C.Musili and C.S.Seshadri, *Geometry of* G/P *- III*, Proc. Ind. Acad. Sci., 87A (1978), 93-177.

[11] V. Lakshmibai, C.Musili and C.S.Seshadri, *Geometry of* G/P *- IV*, Proc. Ind. Acad. Sci., 88A (1979), 279-362.

[12] V. lakshmibai, C.Musili and C.S.Seshadri, *Geometry of* G/P, Bull. A.M.S., (1979), 432-435.

[13] V. Lakshmibai and K.N.Rajeswari, *Towards a standard monomial theory for exceptional groups*, Contemporary Math., A.M.S., Vol. 88 , 1989.

[14] l-r-s V. Lakshmibai, V. Ravikumar snd W. Slofstra, *Cotangent bundle to Cominuscule Grassmannian varieties*, Michigan Math. J. 65 (2016), 749-759.

[15] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in* SL_n/B, Proc. Ind. Acad. Sci. 1990, 45-52.

[16] V. Lakshmibai and C.S.Seshadri, *Geometry of* G/P *- II*, Proc. Ind. Acad. Sci., 87A (1978), 1-54 .

[17] V. Lakshmibai and C.S.Seshadri,*Geometry of* G/P *- V*, J. Alg., Vol. 100 (1986), 463-557.

[18] V. Lakshmibai and C.S.Seshadri, *Singular locus of a Schubert variety*, Bull. A.M.S., Vol. 11 (1984),

[19] V. Lakshmibai and C.S.Seshadri, *Standard monomial theory for* $\hat{SL}_2$, Advanced Studies in Mathematical Physics, Vol. 7 (1989), Proceedings of the conference "Infinite Dimensional Lie Algebras and Groups", held at CIRM, Luminy, Marseille, July 4-8, 1988, p.178-234.

[20] V. Lakshmibai and C.S.Seshadri, *Standard monomial theory and Schubert varieties-a survey*, Proceedings of the Conference on "Algebraic Groups and Applications", 1991, held at Hyderabad, India, 1989, 279-322, Manoj Prakashan.

[21] V. Lakshmibai, C.S. Seshadri and R. Singh, *Cotangent bundle to the Flag variety-I*, Transformation Groups, 2017.

[22] P. Littelmann *Contracting modules and standard monomial theory*, JAMS. 11, 551-567 (1998).
[23] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990), 447-498.
[24] C.S. Seshadri, *Geometry of G/P-I*, in "C. P. Ramanujan: A Tribute, 207-239, Published for the Tata Institute of Fundamental Research, Bombay, Springer-Verlag, Berlin/New York, 1978.
[25] E. Strickland, *On the conormal bundle of the determinantal variety*, J. Algebra 75 (1982), 523-537.

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