C.S.SESHADRI
A MATHEMATICAL LUMINARY

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1. *Introduction*

Conjeevaram Srirangachari Seshadri was a mathematical luminary of the 20th Century in post-independent India. Seshadri was born on February 29th, 1932, in Kanchipuram. He was the eldest among twelve children of his parents, Sri C. Srirangachari (a well-known advocate in Chengleput, a town 60 kms south of Chennai) and Srimati Chudamani. Seshadri's entire schooling was in Chengleput. He joined the Loyola College, Chennai in 1948, and he graduated from there in 1953 with a BA (Hons) degree in mathematics. Seshadri married Sundari in 1962.

During his years at college, Prof. S.Narayanan and Fr. C. Racine played a decisive role in Seshadri's taking up mathematics as a profession.

Seshadri joined the Tata Institute of Fundamental Research, Mumbai in 1953 as a student. He received his Ph.D. degree in 1958 from the Bombay University for his thesis entitled "Generalised multiplicative meromorphic functions on a complex manifold". His thesis adviser was Professor K. Chandrasekaran who shaped the mathematical career of Seshadri as he did for many others.

Seshadri spent the years 1957-60 in Paris, where he came under the influence of many great mathematicians of the French school, like Chevalley, Cartan, Schwartz, Grothendieck and Serre.

He returned to the TIFR in 1960 and was a member of the faculty of the School of Mathematics until 1984, where he was responsible for establishing an active school of algebraic geometry. He moved to the Institute of Mathematical Sciences, Chennai in 1984.

In 1989, Seshadri became the director of the Chennai Mathematical Institute, then called the SPIC Mathematical Institute, founded by A.C. Muthiah.

Seshadri is a recipient of numerous distinctions. He received the Bhatnagar Prize in 1972 and was elected a fellow of the Royal Society, London in 1988. He has held distinguished positions in various centres of mathematics, all over the world. In 2006, Seshadri was awarded the TWAS Science Prize along with Jacob Palis for his distinguished contributions to science.

In the past five years since he received the National Professorship, Seshadri has been awarded the H.K. Firodia Award for Excellence in Science and Technology, Pune, 2008, the Rathindra Puraskar from Shantiniketan's Visva-Bharati University, Kolkata, 2008,the Padma Bhushan by the President of India, 2009. More recently, he was elected a Foreign Associate of the US National Academy of Sciences, 2010. In 2013, Seshadri was awarded Docteur Honoris Causa of the Université Pierre et Marie Curie in Paris.

He passed away on the 17th July in his home in Mandaveli, Chennai [1]. He is survived by his sons Narasimhan and Giridhar and four grandchildren Sanjana, Rangasai, Dev and Anant. Seshadri had been suffering from Parkinson's for the past several months and after the passing of his wife Sundari in October 2019, his condition had been deteriorating.

2. *Seshadri, the person*

I have known Seshadri as his doctoral student since 1984 and later as his collaborator and colleague at the Chennai Mathematical Institute. I vividly remember his lectures. Notes were prepared with utmost meticulousness and the talks were quite spartan but always insightful. Every lecture had something as a take-away for an aspiring researcher. Getting praise from him was something of a rarity. This used to come only as an award for

[1] He was with his family when this happened. I dedicate this article to his family, his son Giridhar, daughter-in-law Padma, grandsons Ranga Sai and Anant and Seshadri-Sundari's daughter-helper over the years, Vadivu, whose devotion to both of them was exemplary.

something which he considered insightful and this was hard to come by for most of us. Many years later, when I was in my forties, after I felt I had done something really significant, he came up to me and said *"There is meat in your work. Now I can say you are a mathematician!"*.

I have witnessed his personality from close quarters for over three decades. As a mathematical personality, I saw someone unique in his vision and insight, an uncanny ability to consistently strike gold in a vast world of mathematics. He was extraordinarily generous with his ideas and shared his insights with one and all and this extreme generosity was his human side as well. His only caveat was that the listeners go back and pursue the ideas to the best of their abilities. There was a complete awareness of his own stature while being modest and humble at the same time. A unique sense of humour and sympathy was the hallmark of his personality, with interests ranging widely from mathematics, philosophy, politics and music. He was confident of his insights and this made him unperturbed during several moments of crisis that the institute faced. I quote Professor K Chandrasekharan, who in a letter to Seshadri on 10 Feb 2013 writes “I cherish the values that inspired the creation of CMI and your unswerving commitment to those values". Seshadri will be remembered for these values.

Seshadri was also an accomplished exponent of the Carnatic Music and till a few days before his passing, he continued to share his musical knowledge and insights with a young musical student Maitreyi from CMI. Seshadri was trained by his grandmother who herself was a student of Nainapillai. Seshadri showed the same traits in his musical discipline as in his mathematical ones. He meticulously did *riyaaz* and his repertoire in Muthuswamy Dikshitar's kritis and Shyama Sastry's kritis was noteworthy. I have had several occasions of listening to his music which can be described as a *royal gait profoundly suited to expressing Dikshitar's kritis.* While singing, a distinctly spiritual side of his used to come to the fore. By a spiritual side, I do not mean anything religious, but a musical one which bore the stamp of an immense *sadhana*, where every nuance was expressed with a spiritual feeling which was way beyond religious emotion.

I close with lines from from W.H. Auden (Hymn to the United Nations):

"Like music when
Begotten notes, New notes beget.
Making the flowing of time a growing.
T'is what it could be....
When even sadness, Is a form of gladness.

3. Seshadri's mathematics in brief outline

Seshadri's doctoral thesis entitled "Generalised multiplicative meromorphic functions on a complex analytic manifold" gave an independent proof of the so-called Birkhoff-Grothendieck theorem on vector bundles on the projective line.

Historically: Grothendieck proved this for arbitrary principal bundles, while the result on vector bundles goes back to Dedekind, R. and H. Weber (1882). Theorie der algebraischen Functionen einer Veränderlichen. J. reine und angew. Math. 92, 181–290.[2]

Seshadri shot to fame early during his visit to Paris. Serre posed the following problem: Over n-dimensional affine space, are there non-trivial vector bundles? In other words, is the following statement true? Is any projective Noetherian module over $K[T_1,...,T_n]$, where K is a field, free? For n = 1, the ring $K[T]$ is an integral principal ideal ring. Therefore, any Noetherian torsion-free $K[T]$-module (in particular, any projective Noetherian module) is free (see for example Lang's Algebra). For n = 2, there are no non-trivial bundles, either and this theorem is due to Seshadri.

The affirmative answer in all dimensions (Any projective Noetherian $K[T_1,...,T_n]$-module is free) is due to A. Suslin and D. Quillen which was done independently by them in the 1970's. Much work arose inspired by Seshadri's ideas, beginning with the work of Pavaman Murthy (Seshadri's first doctoral student) leading to a large body of very impressive work from TIFR.

Seshadri worked broadly in an area of mathematics called Algebraic Geometry. Around the time that he finished his doctoral work, the subject itself was undergoing a unique revolution in the hands of a French mathematical giant by name Alexandre Grothendieck. Seshadri went to Paris in 1957 and very quickly entered the

[2]http://wwwmath.uni-muenster.de/u/scharlau/scharlau/grothendieck/Grothendieck.pdf for a nice historical outline.

sanctum of this new temple of algebraic geometry. This provided a distinctively unifying perspective which connected it to all branches of mathematics at some level. André Weil had published striking conjectures linking number theory to topology, two very distinct branches of mathematics and this was one of the driving forces behind the renaissance in Algebraic Geometry in the hands of Serre-Grothendieck and several others. Algebraic Geometry was somewhat mysterious in that it provided a wonderful synthesis of ideas by the process of providing a powerful language for the expression. Seshadri gave three talks in the Chevalley seminar one on Picard varieties and their compactifications and one on Cartier operations.

It is in this setting that one could view one of Seshadri's deepest researches, which began in collaboration with his friend and colleague M.S. Narasimhan. This work had its roots in the work of André Weil on "generalization of Abelian functions" (1938) and its foundations were closely linked to the work of Poincaré on the so-called "uniformization theorem". I quote from the paper of Atiyah-Bott (1982)

> The connection between holomorphic and unitary structures was already apparent in Weil's paper, and in the classical case of line bundles it is essentially equivalent to the identification between holomorphic and harmonic 1-forms, which in turn was the starting point for Hodge's general theory of harmonic forms.

The Narasimhan-Seshadri theorem sets up a correspondence between two basic classes of objects, namely "irreducible unitary representations of fundamental groups of Riemann surfaces" and "a class of geometric objects called vector bundles on algebraic curves". The first class of objects was what could be termed "topological" and the second "algebro-geometric". Let me be a bit more precise.

The non-abelian version of the Jacobian needed to tackle several basic obstructions. The first paper of Narasimhan-Seshadri (1964) handled the non-abelian structure of the fundamental group of X. They showed that the space of irreducible unitary representations of the fundamental group of a compact Riemann surface X was naturally a complex manifold.

On the side of bundles, the basic issue was that any decent topology on the space of bundles of a given degree and rank is necessarily *non-separated*. Moreover, such bundles are not “bounded" i.e. they cannot be parametrized by a finite number of varieties. Thus to obtain a “projective moduli space", one has to restrict oneself to a suitable subclass of bundles which would give separatedness.

In early 60's David Mumford had revived and newly built the Geometric Invariant Theory of Hilbert into a superstructure, laying out strategies for construction of “compactifications of moduli problems". One of his examples where he applied GIT was in constructing moduli space of bundles. His GIT naturally gave him the correct subclass and he defined slope-stability of bundles and he constructed moduli of stable vector bundles of degree d and rank r on curves as a quasi-projective variety. Recall that to every vector bundle V on a smooth projective curve, Mumford defines the slope $\mu(V) := \deg(V)/\mathrm{rank}(V)$ and a bundle is Mumford-stable if the slope μ strictly decreases when we restrict to a proper sub-bundle.

Narasimhan and Seshadri showed that irreducible unitary representations of the fundamental group of X correspond precisely to stable bundles of degree 0 on X. These were two of the scripts of a trilingual inscription à la the Rosetta stone, the third came up in the work of Donaldson in 1986. What followed was spectacular. Many subtle and beautiful aspects of differential geometry, topology, mathematical physics and number theory got unravelled miraculously.

They do more, they show how this can be extended to the case when the degree need not be zero. This case was a precursor to "parabolic bundles" which Seshadri later developed along with Mehta. Mehta and Seshadri prove the analogue of the Narasimhan-Seshadri theorem for unitary representations of more general Fuchsian groups by relating these to parabolic bundles on X.

Very recently, in a paper which appeared in 2015, Seshadri and I completed the picture by setting up the correspondence for homomorphisms of these Fuchsian groups to the maximal compact subgroups of semisimple groups. Parahoric torsors are the objects which extend parabolic bundles.

I now take up two papers of Seshadri, the first entitled “Some results on the quotient space by an algebraic group of automorphisms" and the second being “Quotient spaces modulo reductive algebraic groups" to which I will return later. The aspect that I wish to highlight here is somewhat general and does not really require the group to be reductive or even affine.

Question: *Let X be a scheme on which a connected algebraic group acts properly. Then does the geometric quotient X/G exist?*

Recall that a G-morphism $f : X \to Y$ is called a *good quotient* if (1) f is a surjective affine G-invariant morphism, (2) $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$ and (3) f sends closed G-stable subsets to closed subsets and separates disjoint closed G-stable subsets of X. The quotient f is called a *geometric quotient* if it is a good quotient and moreover for each $x \in X$, the G-orbit $G.x$ is closed in X.

It is known that the question as stated above fails in general but Seshadri gave some basic criteria under which it holds. He proved the following theorem: let X be a normal scheme of finite type (or more generally a normal algebraic space of finite type over k) and G a connected affine algebraic group acting properly on X. Then the geometric quotient X/G exists as a normal algebraic space of finite type. When the action is proper, a geometric quotient is simply a topological quotient with the property $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$.

Seshadri developed the important technique of *elimination of finite isotropies* which goes as follows. Let X be an irreducible excellent scheme over k and G affine algebraic group acting properly on X. Then there is a diagram:

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "p"] & X \\ Z & \end{tikzcd}$$

where Y is irreducible and G acts properly on Y. Further, p is a Zariski locally trivial principal G-bundle and q a finite dominant G-morphism with Y/X Galois with Galois group Γ whose action on Y commutes with the G-action. In a sense completing the square is the goal. These ideas are central to the major developments by Kollar and Keel and Mori on “Quotients" in the nineties.

I now come to Seshadri's contributions to Geometric Invariant Theory (GIT). In his paper on “Unitary bundles" Seshardri relates unitary bundles to the natural compactification arising from GIT construction of the moduli space. This is important as much of the subsequent work revolves around the study of compact moduli spaces. Irreducible unitary bundles are the “simple" objects of this theory. Two bundles are S-equivalent if they have the same Jordan-Hölder decomposition. The points of the compact moduli space are then the S-equivalence classes of bundles of degree zero and rank r. The beautiful convergence of two lines of thought about these fundamental concepts of (semi)stable bundles was well expressed by Mumford on the occasion of Seshadri's seventieth birthday:

> “But I guess what thrilled both of us – it certainly thrilled me – was when our work on vector bundles on curves arrived at the same idea from two such different directions. What a strange thing it was that three people (you, me, M.S.) on *opposite* sides of the world (which, by the way, seemed a lot bigger in those days) using *totally* different techniques should construct the *same* compact moduli space".

I will very briefly touch on a few other papers of Seshadri in the subject of GIT. This will give a feeling for the breadth and depth of his contributions. The first one was Mumford's conjecture for $GL(2)$ which apart from proving the conjecture gave a restricted “valuative criterion" which predates the famous Langton criterion. This approach of Seshadri's became the standard prototype for all moduli constructions, the most general one being the one by Simpson in the early nineties.

Let me define geometric reductivity of a group G. *Let G be a reductive algebraic group over an algebraically closed field k. Then G is* GEOMETRICALLY REDUCTIVE *if, for every finite-dimensional rational G-module V and a G-invariant point $v \in V$, $v \neq 0$, there is a G-invariant homogeneous polynomial F on V of positive degree such that $F(v) \neq 0$.*

Mumford's conjecture says the following: *Reductive algebraic groups are geometrically reductive.* This was first proved for the case of $SL(2)$ (hence $GL(2)$) in characteristic 2 by Tadao Oda, and in all characteristics by Seshadri. W. Haboush proved the conjecture for a general reductive G in the 1974. Haboush's proof uses in an essential way the irreducibility of the Steinberg representation. A germ of this idea can perhaps be traced back to the appendix to Seshadri's paper by Raghunathan!

There is also a different approach to the problem due to Formanek and Procesi, à priori for the full linear group , but the general case can be deduced from this. Seshadri in the late 70's finally extended geometric reductivity over general excellent rings which is a basic tool for constructing moduli in mixed characteristics.

Seshadri's paper on “Quotients modulo reductive groups" which has already been referred to before, has several beautiful ideas. He introduces the notion of “G-properness" which under some simple conditions shows that quotients, if they exist, are "proper and separated". One of the basic results in this paper is the following: *Let X be a projective variety on which there is given an action of a reductive algebraic group G with respect to an ample line bundle L on X. Let X^{ss} and X^s denote respectively the semi-stable locus and the stable locus of the action of G on (X, L). Suppose that X is normal, $X^{ss} = X^s$, and G acts freely on X. Then the geometric quotient X^s/G exists as a normal projective variety. Loosely put, this is Mumford's conjecture when "semistable = stable".* Seshadri then gives a general technique to ensure the condition $X^{ss} = X^s$ can be made to hold. These have played a central role in several subsequent developments.

Seshadri (in the late 60's) wanted to prove the general Mumford conjecture using the geometric approach which was roughly equivalent to showing that the set Y of equivalence classes of semi-stable points for a linear action of G on a projective scheme X has a canonical structure of a projective scheme. The first difficulty is getting a natural scheme theoretic structure on Y. The second one, more difficult is to prove its projectivity. When "stable = semi-stable" Seshadri showed that Y is a proper scheme and the proof reduces to checking the Nakai-Moishezon criterion for L on Y. *This process led to Seshadri's ampleness criterion and Seshadri constants.*

Around 2009, Seshadri and Pramath Sastry completed Seshadri's old argument. The key new ingredient (work of Sean Keel) was to be able to prove that under some conditions, line bundles which are "nef" and "big" are semi-ample. It was a recursive property for "nef" line bundles to become semi-ample, in a sense a "Nakai-Moisezon" for semi-ampleness.

I now turn to give a very brief account of his work on standard monomial theory much of which in its later developments was a collaboration with V. Lakshmibai and C. Musili. The modern standard monomial theory was initiated by C.S. Seshadri in the early 1970's which was a vast generalisation of the classical theory of Hodge for the Grassmannians.

The broad aim of this theory was the construction of bases for the space of sections of line bundles on Schubert varieties which reflects the intrinsic geometry of the Schubert variety and the intricate combinatorics of the Weyl group. The theory has led to very fundamental developments in the fields of Representation theory, Geometry and Combinatorics.

Following a series of basic papers written in collaboration with V. Lakshmibai and being guided by careful analysis and a study of Schubert varieties for exceptional groups, Lakshmibai and Seshadri formulated the LS conjectures. The key point of the conjectures was that it gave an indexing of the SMT bases which implied a remarkable character formula now termed the LS character formula.

There was a second aspect to these conjectures which constructed bases for the usual Demazure modules associated to the Schubert varieties. Peter Littelmann proved these conjectures by bringing in fresh inputs and new ideas from the theory of Quantum groups.

4. *Seshadri's contribution to mathematics education*

The Chennai Mathematical Institute in its present form was founded in 1998 but its roots go back to 1989 when Seshadri founded a new institute, then called the School of Mathematics, SPIC Science Foundation. The Chennai Mathematical Institute (CMI) is an unique institution in India which attempts to integrate undergraduate education with research; it grew out of Seshadri's vision that higher learning can be had only in an atmosphere of active research amidst the presence of masters in the subject. It was a brave venture in the face of extraordinary opposition and skepticism even from his very close friends and well-wishers. It was his dream to build a center of learning which can compare itself with the great centers such as the École Normale in Paris, the Oxford and Cambridge Universities in England and the Harvard University in the U.S. It opens up opportunities for the gifted students in India to learn in this unique academic atmosphere and also gives possibilities for the active researchers to participate in this experiment which one believes will leave an everlasting influence on the development of mathematics in India.

It would not be an exaggeration to say that the Chennai Mathematical Institute is now rated as one of the best schools in the world for under-graduate studies in mathematics. This is indeed a first big step in its stride and much still needs to be done to fulfill Seshadri's dream.