CMI BSc entrance exam on May 19, 2024

Unless specified otherwise, in this exam all numbers are real and "function" means a function whose codomain as well as domain is the set of all real numbers or an implied subset.

Instructions for Part A

- **Part A is worth 40 points.** It has 21 independently graded questions, which are numbered 2 to 22 for technical reasons. *There is no negative marking.*
- For each of the 21 Part A questions, **type in your answer** as directed.
  - In multiple choice questions, type the label(s) of the correct option(s) from the given list of options. There are no True/False questions.
  - For numerical answers, unless specified otherwise, the answer is an integer which you should enter in its ordinary base 10 representation.
  - In remaining cases, the format of the answer is explained with the question and an example is given in **blue**. *Read the instructions carefully.*
- **Part A will be used for screening.** Part B is assured to be graded if you meet *any one* of the following two conditions. (i) You score at least 24 in part A. (ii) You are among the top 400 students in part A. Thus part B will be graded for at least 400 students, more if enough students score at least 24 in part A.

Instructions for Part B

- **Part B has 6 problems worth a total of 80 points.** Solve each part B problem on the designated pages in the answer booklet.
- **Clearly explain your entire reasoning**. No credit will be given without correct reasoning. You may solve a later part of a problem by assuming some previous part(s), even if you could not do the earlier part(s).
- You are advised to spend at least 2 hours on part B.

Part A questions

Information for questions (2) and (3) (There was no question 1 for technical reasons.)

A test developed to detect Covid gives the correct diagnosis for 99% of people with Covid. It also gives the correct diagnosis for 99% of people without Covid. In a city $\frac{1}{1000}$ of the population has Covid. Answer questions (2) and (3) as per the instruction below.

Instruction for (2) and (3)

If the probability is $x\%$, then your answer should be the integer closest to x. E.g., for probability $\frac{1}{3} = 33.33\ldots\%$, you should type 33 as your answer. For probability $\frac{2}{3}$ you should type 67 as your answer.

Questions

(2) What is the probability that a randomly selected person tests positive? (We assume that in our random selection every person is equally likely to be chosen.) [2 points]

(3) Suppose that a randomly selected person tested positive. What is the probability that this person has Covid? [2 points]

Information for questions (4) (5) and (6)

Consider the polynomial

$$p(x) = x^6 + 10x^5 + 11x^4 + 12x^3 + 13x^2 - 12x - 11.$$

Questions

(4) Find the remainder when $p(x)$ is divided by $x + 1$. [1 point]

(5) Let $z_1, z_2, z_3, z_4, z_5, z_6$ be the six complex roots of $p(x)$. Evaluate $\sum_{i=1}^{6} z_i^2$. [2 points]

(6) Find an integer n with the least possible absolute value such that $p(x)$ has a real root between n and $n+1$. Write this number *along with your reason* as per the given instruction. [2 points]

Instruction for (6): Write two numbers separated by a comma: value of n, number of the theorem below that justifies this answer. E.g., if you think that $n = 5$ because of the factor theorem, then type **5,1** as your answer with no space, full stop or any other punctuation.

1. Factor theorem
2. Mean value theorem
3. Intermediate value theorem
4. Fundamental theorem of algebra
5. Fundamental theorem of calculus

Information for questions (7) and (8)

Two mighty frogs jump *once per unit time* on the number line as described in the question.

Questions (7) The first frog is at $x = 2^i$ at time $t = i$. How many numbers of the form $7n+1$ (with n an integer) does the frog visit from $t = 0$ to $t = 99$ (both endpoints included)? [3 points]

(8) The second frog starts at $x = 0$ and jumps $i+1$ steps to the right just after $t = i$, so that at times $t = 0, 1, 2, 3, \ldots$ this frog is at positions $x = 0, 1, 3, 6, \ldots$ respectively. How many numbers of the form $7n+1$ (with n an integer) does the frog visit from $t = 0$ to $t = 99$ (both endpoints included)? [3 points]

Information for questions (9) (10) (11)

Let $O = (0,0,0)$, $P = (19, 5, 2024)$ and $Q = (x, y, z)$ be points in 3-dimensional space where Q is an unknown point.

Consider vector $\mathbf{u} = \overrightarrow{OP} = 19\,\hat{\imath} + 5\,\hat{\jmath} + 2024\,\hat{k}$ and unknown vector $\mathbf{v} = \overrightarrow{OQ} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$.

Instruction: for the specified set choose the correct option describing it and type in the number of that option. E.g., if you think the given set is a line, enter **3** as your answer with no full stop or any other punctuation.

Questions (9) $\{Q \mid \mathbf{u} \cdot \mathbf{v} = 2024\}$. [1 point] (10) $\{Q \mid \mathbf{u} \cdot \mathbf{v} = -2024\sqrt{\mathbf{v} \cdot \mathbf{v}}\,\}$. [2 points]

(11) $\{Q \mid \mathbf{u} \cdot \mathbf{v} = 2024\,(\mathbf{v} \cdot \mathbf{v})\}$. [2 points]

Options

1. The empty set
2. A singleton set
3. A line
4. A pair of lines
5. A circle
6. A plane perpendicular to **u**
7. A plane parallel to **u**
8. An infinite cone
9. A finite cone
10. A sphere
11. None of the above

Information for questions (12) and (13)

An integer d is called a factor of an integer n if there is an integer q such that $n = qd$. In particular the set of factors of n contains n and contains 1. You are given that $2024 = 8 \times 11 \times 23$.

Questions

(12) Write the number of *even positive integers* that are factors of 2024^2. [2 points]

(13) Write the number of ordered pairs (a, b) of *positive* integers such that $a^2 - b^2 = 2024^2$. If there are infinitely many such pairs, write the word **infinite** as your answer. [3 points]

Information for questions (14) (15) (16)

A good path is a sequence of points in the XY plane such that in each step exactly one of the coordinates increases by 1 and the other stays the same. E.g.,

$$(0, 0), (1, 0), (2, 0), (2, 1), (3, 1), (3, 2), (3, 3)$$

is good path from the origin to (3,3). It is a fact that there are exactly 924 good paths from the origin to (6,6).

Questions

(14) Find the number of good paths from $(0, 0)$ to $(6, 6)$ that pass through both the points $(1, 4)$ and $(2, 3)$. [1 point]

(15) Find the number of good paths from $(0, 0)$ to $(6, 6)$ that pass through both the points $(1, 2)$ and $(3, 4)$. [2 points]

(16) Find the number of good paths from $(0, 0)$ to $(6, 6)$ such that *neither* of the two points $(1, 2)$ and $(3, 4)$ occurs on the path, i.e., the path must *miss both* of the points $(1, 2)$ and $(3, 4)$. [3 points]

Information for questions (17) to (20)

Suppose f is a function whose domain is X and codomain is Y. It is given that $|X| > 1$ and $|Y| > 1$. No other information is known about X, Y and f. **Instruction:** Write the number of a *single correct option* for the given statement S.

Questions [1 point each]

(17) S = "For each x in X, there exists y in Y such that $f(x) = y$." [1 point]

(18) S = "For each y in Y, there exists x in X such that $f(x) = y$." [1 point]

(19) S = "There exists a unique x in X such that for each y in Y it is true that $f(x) = y$." [1 point]

(20) S = "There exists a unique y in Y such that for each x in X it is true that $f(x) = y$." [1 point]

Options

1. S is always true.
2. S is always false.
3. S is true if and only if f is one-to-one.
4. If S is true then f is one-to-one but the converse is false.
5. If f is one-to-one then S is true but the converse is false.
6. S is true if and only if f is onto.
7. If S is true then f is onto but the converse is false.
8. If f is onto then S is true but the converse is false.
9. S is true if and only if f is a constant function.
10. If S is true then f is a constant function but the converse is false.
11. If f is a constant function then S is true but the converse is false.
12. None of the above.

Information for questions (21) and (22)

Suppose a differentiable function f from $\mathbb{R}$ to $\mathbb{R}$ has a local maximum at $(a, f(a))$ (This means there are numbers m and M such that (i) $m < a < M$ and (ii) $f(a) \geq f(x)$ for any $x \in [m, M]$.) The proof of a standard result is sketched below. Complete it as instructed.

Proof: For sufficiently __1__ $h > 0$, it is given that $f(a+h)$ __2__ __3__.

Therefore for such h the quantity __4__ must be __5__ __6__.

By taking the limit of this quantity as $h \to 0$ from the right, we get that __7__ must be __8__ __9__.

A parallel argument for suitable negative values of h gives that __10__ must be __11__ __12__.

Combining both conclusions gives the desired result: __13__ __14__ __15__. Note that the mentioned limits exist because __16__.

Questions

(21) Write a sequence of 9 letters indicating the correct options to fill in the numbered blanks 1 to 9. Do not use any spaces, full stop or other punctuation. E.g., **ABACDIJKB** is in the correct format. [3 points]

(22) Write a sequence of 7 letters indicating the correct options to fill in the numbered blanks 10 to 16. [2 points]

Options

A. small

B. large

C. $\geq$

D. $>$

E. $\leq$

F. $<$

G. $=$

H. $\neq$

I. 0

J. $f(a)$

K. $\frac{f(a+h)-f(a)}{h}$

L. $f'(a)$

M. f is differentiable

N. f is continuous

Part B Problems for BSc entrance exam on May 19, 2024

B1. [10 points] (a) Draw a qualitatively accurate sketch of the unique bounded region R in the *first quadrant* that has *maximum possible* finite area with boundary described as follows. R is bounded below by the graph of $y = x^2 - x^3$, bounded above by the graph of an equation of the form $y = kx$ (where k is some constant), and R is *entirely enclosed by the two given graphs*, i.e., the boundary of the region R must be *a subset* of the union of the given two graphs (so R does not have any points on its boundary that are not on these two graphs). Clearly mark the relevant point(s) on the boundary where the two given graphs meet and **write the coordinates of every such point**.

(b) Consider the solid obtained by rotating the above region R around Y-axis. Show how to find the volume of this solid by doing the following: Carefully set up the calculation with justification. Do enough work with the resulting expression to reach a stage where the final numerical answer can be found mechanically by using standard symbolic formulas of algebra and/or calculus and substituting known values in them. Do not carry out the mechanical work to get the final numerical answer.

B2. [15 points] (a) Find the domain of the function $g(x)$ defined by the following formula.

$$g(x) = \int_{10}^{x} \log_{10}\left(\log_{10}(t^2 - 1000t + 10^{1000})\right) dt.$$

Calculate the quantities below. You may give an approximate answer where necessary, but clearly state which answers are exact and which are approximations.

(b) $g(1000)$.

(c) x in [10,1000] where $g(x)$ has the maximum possible slope.

(d) x in [10,1000] where $g(x)$ has the least possible slope.

(e) $\lim_{x\to\infty} \frac{\ln(x)}{g(x)}$ if it exists.

B3. [15 points] (a) For non-negative numbers a, b, c and any positive real number r prove the following inequality *and* state precisely when equality is achieved.

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$

Hint: Assuming $a \geq b \geq c$ do algebra with just the first two terms. What about the third term? What if the assumption is not true?

(b) As a special case obtain an inequality with $a^4 + b^4 + c^4 + abc(a+b+c)$ on one side.

(c) Show that if $abc = 1$ for positive numbers a, b, c, then

$$a^4 + b^4 + c^4 + a^3 + b^3 + c^3 + a + b + c \geq \frac{a^2+b^2}{c} + \frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + 3.$$

B4. [**10 points**] Find all solutions of the following equation where it is required that x, k, y, n are *positive integers* with the exponents k and n both > 1.

$$20x^k + 24y^n = 2024.$$

B5. [**15 points**] (a) Find all complex solutions of $z^6 = z + \overline{z}$.

(b) For an integer $n > 1$, how many complex solutions does $z^n = z + \overline{z}$ have?

B6. [**15 points**] A list of k elements, possibly with repeats, is given. The goal is to find if there is a *majority element*. This is defined to be an element x such that the number of times x occurs in the list is *strictly* greater than $\frac{k}{2}$. (Note that there need not be such an element, but if it is there, it must be unique.) A celebrated efficient way to do this task uses two functions f and m with domain $\{1, 2, \ldots, k\}$. The functions are defined inductively as follows.

Define $f(1) =$ first element of the list, $m(1) = 1$.

Assuming f and m are defined for all inputs from 1 to i, define

$$f(i+1) = \begin{cases} f(i) & \text{if } m(i) > 0 \\ (i+1)^{th} \text{ element of the list} & \text{if } m(i) = 0 \end{cases}$$

$$m(i+1) = \begin{cases} m(i) - 1 & \text{if } m(i) > 0 \text{ and } (i+1)^{th} \text{ element of the list is other than } f(i) \\ m(i) + 1 & \text{otherwise} \end{cases}$$

(a) For the example of length 15 given below, write a sequence of 15 letters showing the values of $f(i)$ and a sequence of 15 numbers directly underneath showing the values of $m(i)$ for $i = 1, 2, \ldots, 15$.

a a b a b c c b b b a b b c b

(b) Prove that in general the list can be divided into two disjoint parts A and B such that

- Part A contains $m(k)$ elements of the list each of which is $f(k)$.
- Part B contains the remaining $k - m(k)$ elements of the list and B can be written as disjoint union of pairs such that the two elements in each pair are distinct.

(c) If there is a majority element, show that it must be $f(k)$. You may assume part (b) even if you did not do it.

(d) Assuming $f(k)$ is the majority element, answer the following two questions. Show by examples that the number of occurrences of $f(k)$ in the list does not determine the value of $m(k)$. Can the value of $m(k)$ be anything in $\{0, \ldots, k\}$? Find constraints if any on the possible values of $m(k)$.

(e) Now assume instead that an element occurs exactly $\frac{k}{2}$ times in the list. Is it necessary that $f(k)$ is such an element?