CHENNAI MATHEMATICAL INSTITUTE
Undergraduate Entrance Examination, 7$^{\text{th}}$ May 2023
Time: 3.5 hours

Unless specified otherwise, in this exam all numbers are real, the domain of each function is the set of real numbers (or an implied subset) and the codomain is also the set of real numbers. You may use the following information wherever you find it relevant.

- $2023 = 17^2 \times 7$.
- One can use long division to find the *gcd* of two positive integers a, b (defined to be a common divisor d of a and b such that d is divisible by any other such common divisor).
- The same procedure stays valid for finding *gcd* of polynomials in one variable with rational/real/complex coefficients.
- Any nonzero polynomial of degree n has n complex roots counting multiplicity.

Part A instructions

- **Part A has 10 questions, each worth 4 points, for a total of 40.** Points for part A will be given based only on the answers you enter into the computer.
- **7 out of the 10 questions are objective**, each with a group of four statements. (These statements are numbered 1 to 28 for technical reasons.) For each statement, *independently* choose one of the three options True / False / No Attempt. In particular there is no guarantee that at least one of the four statements in a given question is true. If you do not choose an option for a statement, it will be treated as No Attempt.
- **Grading scheme for the 7 objective questions** is as follows.

All 4 answers correct	4 points
3 correct and 1 No Attempt	2 points
2 correct and 2 No Attempt	1 point
Anything else	0 points

Note that getting even one of the four answers wrong will result in zero points for that question. So if you are not sure, you are advised to choose No Attempt instead of guessing.

- The remaining three questions have two parts each. For each part, enter only the final answer into the computer in the precise format specified in the question. **There is no negative marking for these three questions.**
- **Part A will be used for screening.** Part B is assured to be graded if you meet *any one* of the following two conditions. (i) You score at least 24 in part A. (ii) You are among the top 400 students in part A. Thus part B will be graded for at least 400 students, more if enough students score at least 24 in part A.

Instrutions for part B $\longrightarrow$

Part B instructions

- **Part B has 6 problems worth a total of 80 points.** See each question for the break-up. You are advised to spend at least 2 hours on part B.
- **Clearly explain your entire reasoning**. No credit will be given without correct reasoning. Partial solutions may get partial credit. You may solve a later part of a problem by assuming a previous part, even if you could not do the earlier part.
- **Solve each part B problem on the designated pages** in the answer booklet. Use the blank pages at the end for rough work OR if you need extra space for any problem. Clearly label any such solution overflowing to last pages. For problems with multiple parts, clearly label your solution to each part separately.

Part A Questions for CMI BSc entrance exam on May 7, 2023

A1. Define the *right derivative* of a function f at $x = a$ to be the following limit if it exists.

$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, where $h \to 0^+$ means h approaches 0 only through positive values.

Statements

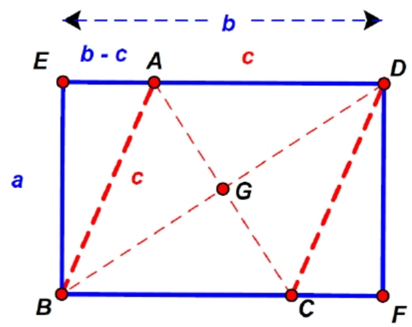
(1) If f is differentiable at $x = a$ then f has a right derivative at $x = a$.

(2) $f(x) = |x|$ has a right derivative at $x = 0$.

(3) If f has a right derivative at $x = a$ then f is continuous at $x = a$.

(4) If f is continuous at $x = a$ then f has a right derivative at $x = a$.

A2. Suppose a rectangle $EBFD$ is given and a rhombus $ABCD$ is inscribed in it so that the point A is on side ED of the rectangle. The diagonals of $ABCD$ intersect at point G. See the indicative figure below.



Statements

(5) Triangles CGD and DFB *must* be similar.

(6) It *must* be true that $\frac{AC}{BD} = \frac{EB}{ED}$.

(7) Triangle CGD *cannot* be similar to triangle AEB.

(8) For *any* given rectangle $EBFD$, a rhombus $ABCD$ as described above can be constructed.

A3. This question is about complex numbers.

Statements

(9) The complex number $(e^3)^i$ lies in the third quadrant.

(10) If $|z_1| - |z_2| = |z_1 + z_2|$ for some complex numbers z_1 and z_2, then z_2 must be 0.

(11) For distinct complex numbers z_1 and z_2, the equation $|(z - z_1)^2| = |(z - z_2)^2|$ has at most 4 solutions.

(12) For each nonzero complex number z, there are more than 100 numbers w such that $w^{2023} = z$.

A4. Statements

(13) $\lim_{x\to 0} e^{\frac{1}{x}} = +\infty$.

(14) The following inequality is true.

$$\lim_{x\to\infty} \frac{\ln x}{x^{100}} < \lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{100}}}.$$

(15) For any positive integer n,

$$\int_{-n}^{n} x^{2023} \cos(nx)dx < \frac{n}{2023}.$$

(16) There is no polynomial $p(x)$ for which there is a single line that is tangent to the graph of $p(x)$ at exactly 100 points.

A5. Statements

(17) $4 < \sqrt{5+5\sqrt{5}}$.

(18) $\log_2 11 < \frac{1+\log_2 61}{2}$.

(19) $(2023)^{2023} < (2023!)^2$.

(20) $92^{100} + 93^{100} < 94^{100}$.

A6. For a sequence a_i of real numbers, we say that $\sum a_i$ converges if $\lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i\right)$ is finite. **In this question all $a_i > 0$.**

Statements

(21) If $\sum a_i$ converges, then $a_i \to 0$ as $i \to \infty$.

(22) If $a_i < \frac{1}{i}$ for all i, then $\sum a_i$ converges.

(23) If $\sum a_i$ converges, then $\sum (-1)^i a_i$ also converges.

(24) If $\sum a_i$ does not converge, then $\sum i \tan(a_i)$ cannot converge.

A7. Statements

(25) To divide an *integer* b by a *nonzero integer* d, define a quotient q and a remainder r to be *integers* such that $b = qd + r$ and $|r| < |d|$. Such integers q and r always exist and are both unique for given b and d.

(26) To divide a *polynomial* $b(x)$ by a *nonzero polynomial* $d(x)$, define a quotient $q(x)$ and a remainder $r(x)$ to be *polynomials* such that $b = qd + r$ and $degree(r) < degree(d)$. (Here $b(x)$ and $d(x)$ have real coefficients and the 0 polynomial is taken to have negative degree by convention.) Such polynomials $q(x)$ and $r(x)$ always exist and are both unique for given $b(x)$ and $d(x)$.

(27) Suppose that in the preceding question $b(x)$ and $d(x)$ have *rational* coefficients. Then $q(x)$ and $r(x)$, if they exist, must also have rational coefficients.

(28) The least *positive* number in the set

$$\{(a \times 2023^{2020}) + (b \times 2020^{2023})\}$$

as a and b range over all *integers* is 3.

A8. You play the following game with three fair dice. (When each one is rolled, any one of the outcomes 1, 2, 3, 4, 5, 6 is equally likely.) In the first round, you roll all three dice. You remove every die that shows 6. If any dice remain, you roll all the remaining dice again in the second round. Again you remove all dice showing 6 and continue.

Questions

(29) Let the probability that you are able to play the second round be $\frac{a}{b}$, where a and b are integers with *gcd* 1. Write the numbers a and b separated by a comma. E.g., for probability $\frac{10}{36}$ you would type 5,18 with no quotations, space, full stop or any other punctuation.

(30) Let the probability that you are able to play the second round but not the third round be $\frac{c}{d}$ where c and d are integers with *gcd* 1. Write only the integer c as your answer. E.g., for probability $\frac{34}{36}$ you would type 17 with no quotations, space, full stop or any other punctuation.

A9. Two lines ℓ_1 and ℓ_2 in 3-dimensional space are given by

$\ell_1 = \{(t-9, -t+7, 6) \mid t \in \mathbb{R}\}$ and $\ell_2 = \{(7, s+3, 3s+4) \mid s \in \mathbb{R}\}$.

Questions

(31) The plane passing through the origin and not intersecting either of ℓ_1 and ℓ_2 has equation $ax + by + cz = d$. Write the value of $|a+b+c+d|$ where a, b, c, d are integers with $gcd = 1$.

(32) Let r be the smallest possible RADIUS of a circle that has a point on ℓ_1 as well as a point on ℓ_2. It is given that r^2 (i.e, the SQUARE of the smallest radius) is an integer. Write the value of r^2.

A10. Consider the part of the graph of $y^2 + x^3 = 15xy$ that is strictly to the right of the Y-axis, i.e., take only the points on the graph with $x > 0$.

Questions

(33) Write the least possible value of y among considered points. If there is no such real number, write NONE (without any spaces or quotation marks or any other punctuation).

(34) Write the largest possible value of y among considered points. If there is no such real number, write NONE (without any spaces or quotation marks or any other punctuation).

B1. [**11 points**] We want to find *odd* integers $n > 1$ for which n is a factor of $2023^n - 1$.

(a) Find the two smallest such integers.
(b) Prove that there are infinitely many such integers.

B2. [**12 points**] Let $\mathbb{Z}^+$ denote the set of *positive* integers. We want to find *all* functions $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following equation holds for any m, n in $\mathbb{Z}^+$.

$$g(n+m) = g(n) + nm(n+m) + g(m).$$

Prove that $g(n)$ must be of the form $\sum_{i=0}^{d} c_i n^i$ and find the precise necessary and sufficient condition(s) on d and on the coefficients $c_0, \ldots, c_d$ for g to satisfy the required equation.

B3. [**13 points**] Suppose that for a given polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, there is exactly one *real* number r such that $p(r) = 0$.

(a) If a, b, c, d are rational, show that r must be rational.
(b) If a, b, c, d are integers, show that r must be an integer.

Possible hint: Also consider the roots of the derivative $p'(x)$.

B4. [**14 points**] There are n students in a class and no two of them have the same height. The students stand in a line, one behind another, in no particular order of their heights.

(a) How many different orders are there in which the shortest student is not in the first position *and* the tallest student is not in the last position?

(b) The *badness* of an ordering is the largest number k with the following property. There is at least one student X such that there are k students taller than X standing ahead of X. Find a formula for $g_k(n)$ = number of orderings of n students with badness k.

Example: The ordering 64 61 67 63 62 66 65 (the numbers denote heights) has badness 3 as the student with height 62 has three taller students (with heights 64, 67 and 63) standing ahead in the line and nobody has more than 3 taller students standing ahead.

Possible hints for (b): It may be useful to first count orderings of badness 1 and/or to find $f_k(n)$ = the number of orderings of n students with badness *less than or equal to* k.

B5. **[15 points]** Throughout this question every mentioned function is required to be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. The symbol $\circ$ denotes composition of functions.

(a) Suppose $f \circ f = f$. Then for each x, one must have $f'(x) =$ _____ or $f'(f(x)) =$ _____. Complete the sentence and justify.

(b) For a non-constant f satisfying $f \circ f = f$, it is known and you may assume that the range of f must have one of the following forms: $\mathbb{R}, (-\infty, b], [a, \infty)$ or $[a, b]$. Show that in fact the range must be all of $\mathbb{R}$ and deduce that there is a unique such function f. (Possible hints: For each y in the range of f, what can you say about $f(y)$? If the range has a maximum element b what can you say about the derivative of f?)

(c) Suppose that $g \circ g \circ g = g$ and that $g \circ g$ is a non-constant function. Show that g must be onto, g must be strictly increasing or strictly decreasing and that there is a unique such increasing g.

B6. **[15 points]** Starting with any given positive integer $a > 1$ the following game is played. If a is a perfect square, take its square root. Otherwise take $a + 3$. Repeat the procedure with the new positive integer (i.e., with $\sqrt{a}$ or $a + 3$ depending on the case). The resulting set of numbers is called the trajectory of a. For example the set $\{3, 6, 9\}$ is a trajectory: it is the trajectory of each of its members.

Which numbers have a finite trajectory? Possible hint: Find the set

$$\{n \mid n \text{ is the smallest number in some trajectory } S\}.$$

If you wish, you can get partial credit by solving the following simpler questions.

(a) Show that there is no trajectory of cardinality 1 or 2.
(b) Show that $\{3, 6, 9\}$ is the only trajectory of cardinality 3.
(c) Show that for any integer $k \geq 3$, there is a trajectory of cardinality k.
(d) Find an infinite trajectory.