CHENNAI MATHEMATICAL INSTITUTE
Undergraduate Entrance Examination, May 2022
Time: 3.5 hours

Unless specified otherwise, in this exam all numbers are real and "function" means a function whose codomain as well as domain is the set of all real numbers (or an implied subset).

Part A instructions

- **Part A has 10 questions, each worth 4 points, for a total of 40.** Every question has a group of four statements. (These statements are numbered 1 to 40 for technical reasons.) For each statement, *independently* choose one of the three given options: True / False / No Attempt. In particular there is no guarantee that at least one of the four statements in a given question is true.

- **Grading scheme.** Each question will be graded as follows.

All 4 answers correct	4 points
3 correct and 1 No Attempt	2 points
2 correct and 2 No Attempt	1 point
Anything else	0 points

Note that getting even one of the four answers wrong will result in zero points for that question. So if you are not sure, you are advised to choose No Attempt instead of guessing.

- **Enter your answers to part A into the computer.** Points for part A will be given based only on answers entered into the computer. If you do not choose an option for a statement, it will be treated as No Attempt.

- **Part A will be used for screening.** Part B is assured to be graded if you meet *any one* of the following two conditions. (i) You score at least 24 in part A. (ii) You are among the top 400 students in part A. Thus part B will be graded for at least 400 students, more if enough students score at least 24 in part A.

Part B instructions

- **Part B has 6 problems worth a total of 80 points.** See each question for the break-up. You are advised to spend at least 2 hours on part B.

- **Clearly explain your entire reasoning**. No credit will be given without correct reasoning. Partial solutions may get partial credit. You may solve a later part of a problem by assuming a previous part, even if you could not do the earlier part.

- **Solve each part B problem on the designated pages** in the answer booklet. Use the blank pages at the end for rough work OR if you need extra space for any problem. Clearly label any such solution overflowing to last pages. For problems with multiple parts, clearly label your solution to each part separately.

CMI BSc entrance make-up exam on May 23, 2022

Part A. Select True or False or No Attempt for each statement.

A1. The three sides of triangle $a < b < c$ are in arithmetic progression (AP) with common difference $d = b - a = c - b$. Denote the angles opposite to sides a, b, c respectively by A, B, C.

Statements

(1) d must be less than a.

(2) d can be any positive number less than a.

(3) The numbers $\sin A, \sin B, \sin C$ are in AP.

(4) The numbers $\cos A, \cos B, \cos C$ are in AP.

A2. You are asked to take three *distinct* points $1, \omega_1$ and ω_2 in the complex plane such that $|\omega_1| = |\omega_2| = 1$. Consider the triangle T formed by the complex numbers $1, \omega_1$ and ω_2.

Statements

(5) There is exactly one such triangle T that is equilateral.

(6) There are exactly two such triangles T that are right angled isosceles.

(7) If $\omega_1 + \omega_2$ is real, the triangle T must be isosceles.

(8) For *any* nonzero complex number z, the numbers $z, z\omega_1$ and $z\omega_2$ form a triangle that is similar to the triangle T.

A3. M is a 3×3 matrix with integer entries. For M we have

(Sum of column 2) $= 4 \times$ (sum of column 1). (Sum of column 3) $= 4 \times$ (sum of column 2).

(Sum of row 2) $= 6 +$ (sum of row 1).
(Sum of row 3) $= 6 +$ (sum of row 2).

Statements

(9) The sum of all the entries in M must be divisible by 21.

(10) None of the row sums is divisible by 7.

(11) One of the column sums must be divisible by 7.

(12) None of the column sums is divisible by 6.

A4. Statements

(13) As $x \to -\infty$ the function $\cos(e^x)$ tends to a finite limit.

(14) As $x \to \infty$ the function $\cos(e^x)$ changes sign infinitely many times.

(15) As $x \to \infty$, the function $\sin(\ln(x))$ tends to a finite limit.

(16) $\sin(\ln(x))$ changes sign only finitely many times as x goes towards 0 from 1.

A5. Statements

(17) $\sqrt[4]{4} < \sqrt[5]{5}$.

(18) $\log_{10} 11 > \log_{11} 12$.

(19) $\frac{\pi}{4} < \sqrt{2 - \sqrt{2}}$.

(20) $(2022!)^2 > 2022^{2022}$.

A6. Let $f(x) = |\frac{\sin x}{x}|^{1.001}$ for $x \neq 0$ and $f(0) = L$ such that f is continuous. Let $I(x) = \int_0^x f(t)dt$.

Statements

(21) $L = 1.001$

(22) $I(0.001) > 0.001$.

(23) As $x \to \infty$ the limit of $I(x)$ is greater than 1001 (possibly ∞).

(24) The function $I(x)$ is *NOT* differentiable at infinitely many points.

A7. Statements

(25) There is a unique natural number n such that $n^2 + 19n - n! = 0$.

(26) There are infinitely many pairs (x, y) of natural numbers satisfying

$$(1 + x!)(1 + y!) = (x + y)!.$$

(27) For any natural number n, consider GCD of $n^2 + 1$ and $(n + 1)^2 + 1$. As n ranges over all natural numbers, the largest possible value of this GCD is 5.

(28) If n is the largest natural number for which $20!$ is divisible by 80^n, then $n \geq 5$.

A8. Let a be a point in the domain of a continuous real valued function f. One says that f has a *flex point* at a if we can find a small interval $(a-\epsilon, a+\epsilon)$ in the domain of f such that the following happens: (i) for all x in the open interval $(a-\epsilon, a)$ the sign of $f''(x)$ is constant and, (ii) for all x in the open interval $(a, a+\epsilon)$ the sign of $f''(x)$ is constant and opposite to the sign of $f''(x)$ in $(a-\epsilon, a)$.

Statements

(29) If f is a cubic polynomial with a local maximum at $x = p$ and a local minimum at $x = q$, then f has a unique flex point at $x = \frac{p+q}{2}$.

(30) If $f''(a) = 0$ then f must have a flex point at a.

(31) Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = -x^2$ for $x < 0$. Then f has no flex points.

(32) If f' is monotonic on an open interval I, then f cannot have a flex point in I.

A9. Suppose A, B and C are three events and $P(A) = a, P(B) = b, P(C) = c$ are known. Let $P(A \cup B \cup C) = p$. The statements below are about whether we can find the value of p if we know some additional information. (Note: $\cup$ is the same as OR. Similarly $\cap$ is the same as AND.)

Statements

(33) We can find the value of p if we know that at least one of a, b, c is 1.

(34) We can find the value of p if we know that at least one of a, b, c is 0.

(35) We can find the value of p if we know that any two of A, B and C are mutually exclusive.

(36) We can find the value of p if we know that any two of A, B and C are independent and we know the value of $P(A \cap B \cap C)$.

A10. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 31 \\ 11 & 22 & k \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where k is a constant and x, y, z are variables.

Statements

(37) Regardless of the value of k, the matrix A is not invertible, i.e., there is no 3×3 matrix B such that $BA =$ the 3×3 identity matrix.

(38) There is a unique k such that determinant of A is 0.

(39) The set of solutions (x, y, z) of the matrix equation $A\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is either a line or a plane containing the origin.

(40) If the equation $A\mathbf{v} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ has a solution, then it must be true that $q = 10p$.

Part B Problems for the make-up exam on Monday, May 23rd

B1. [12 points] Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $L = \{a, b, c\}$.

(i) Suppose we arrange the 12 elements of $L \cup N$ in a line such that no two of the three letters occur consecutively. If the order of the letters among themselves does not matter, find the number such arrangements.

(ii) Find the number of functions from N to L such that exactly 3 numbers are mapped to each of a, b and c.

(iii) Find the number of onto functions from N to L.

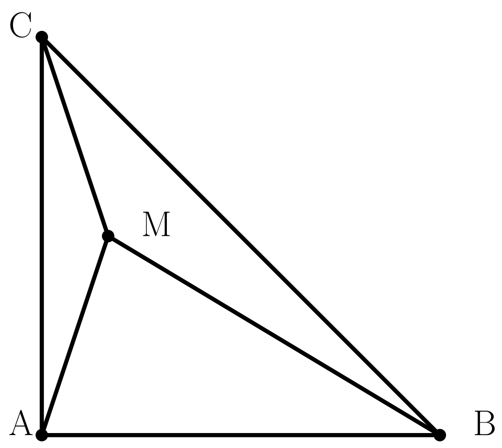
B2. [12 points] Let f be a function from natural numbers to natural numbers that satisfies

$$f(n) = n - 2 \quad \text{for } n > 3000;$$
$$f(n) = f(f(n+5)) \quad \text{for } n \leq 3000.$$

Show that $f(2022)$ is uniquely decided and find its value.

B3. [14 points] In $\triangle ABC$, $\angle BAC = 2\angle ACB$ and $0° < \angle BAC < 120°$. A point M is chosen in the interior of $\triangle ABC$ such that $BA = BM$ and $MA = MC$. Prove that $\angle MCB = 30°$. See the schematic figure below (NOT to scale).

Hint (use this or your own method): Draw a suitable segment CD of appropriate length making an appropriate angle with CA.



B4. [**14 points**] We want to find a *nonzero* polynomial $p(x)$ with integer coefficients having the following property.

$$\text{Letting } q(x) := \frac{p(x)}{x(1-x)}\,, \qquad q(x) = q\Big(\frac{1}{1-x}\Big) \text{ for all } x \notin \{0,1\}.$$

(i) Find one such polynomial with the smallest possible degree.

(ii) Find one such polynomial with the largest possible degree OR show that the degree of such polynomials is unbounded.

B5. [**14 points**] Let $\mathbb{R}_+$ denote the set of positive real numbers. A one-to-one and onto function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *golden* if $f'(x) = f^{-1}(x)$ for every $x \in \mathbb{R}_+$.

(i) Find all golden functions (if any) of the form $f(x) = a\,x^b$. Find all golden functions (if any) of the form $f(x) = a\,b^x$. In both cases a and b are suitable real numbers.

(ii) Show that there is no one-to-one and onto function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f^{-1}(x)$ for every $x \in \mathbb{R}$.

B6. [**14 points**] Suppose an integer $n > 1$ is such that $n+1$ is not a multiple of 4 (i.e., such that n is not congruent to 3 mod 4). Prove that there exist $1 \le i < j \le n$ such that the following is a perfect square.

$$\frac{1!\,2!\cdots n!}{i!\,j!}$$

Hint (use this or your own method): Make cases and first treat the case $n = 4k$.