CHENNAI MATHEMATICAL INSTITUTE

Undergraduate Programme in Mathematics and Computer Science/Physics
Solutions of the 23rd May 2022 exam

Note: The solutions below consist only of main steps and strategies and do not contain all the details expected in the exam.

B1. [**12 points**] Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $L = \{a, b, c\}$.

1. Suppose we arrange 12 elements of $L \cup N$ in a line such that no two of the three letters occur consecutively. If the order of the letters among themselves does not matter, find the number of such arrangements.

2. Find the number of functions from N to L such that exactly 3 numbers are mapped to a, b, c.

3. Find the number of onto functions from N to L.

Solution:

1. Imagine that the numbers are placed with a gap between two consecutive numbers. Moreover there is a gap before 1 and one after 9. So there are 10 gaps, where letters can be placed. In order to find the number of such arrangement first choose 3 gaps from 10 and then permute the 9 numbers. The final answer is $\binom{10}{3} \cdot 9!$.

2. Nine digits have to be placed in 3 groups. The answer is $\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3}$.

3. By inclusion-exclusion principal the answer is

$$3^9 - 3 \cdot 2^9 + 3 \cdot 1^9.$$

The first summand is the total number of functions, the second summand is the number of function which miss 2 values and last summand is the number of functions which miss 1 value.

B2. [**12 points**] Let f function from natural numbers to natural numbers that satisfies

$$f(n) = n - 2, \quad \text{for } n > 3000;$$
$$f(n) = f(f(n+5)) \quad \text{for } n \leq 3000.$$

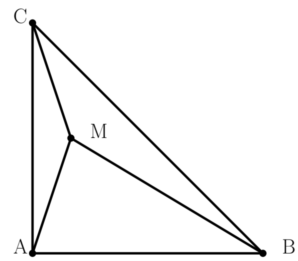
Show that $f(2022)$ is uniquely determined and find its value.

Solution: The important step is to discover the following closed form expression for the function:

$$f(n) = 2999 + (n+2)\text{mod}(3).$$

The equality can be derived in a number of ways, including induction. Once this is established it is straightforward to conclude that $f(2022) = 3001$.

B3. [**14 points**] In $\triangle ABC$, $\angle BAC = 2\angle ACB$ and $0° < \angle BAC < 120°$. A point M is chosen in the interior of $\triangle ABC$ such that $BA = BM$ and $MA = MC$. Prove that $\angle MCB = 30°$. See the schematic figure below (not to scale).



Solution: There are various ways to solve this problem. Let us start with a geometric approach.

Construct a line segment CD such that it is equal to AB and $\angle DCA = \angle BAC$. Denote the intersection of AD and BC by E. First step, show that $\triangle BAC \equiv \triangle DCA$. Second step, show that $\triangle CDE \equiv \triangle ABE$. Next, show that $\triangle EMA \equiv \triangle EMC$. Now conclude that $\angle CAB = 90°$, this will lead to the solution.

Another construction is to draw a line segment BD such that it is equal to CD and is parallel to AC. First show that $ABDC$ is an isosceles trapezium. Use it prove that $\triangle ABM \equiv \triangle CMD$. Conclude that $\triangle BMD$ is equilateral. Use this along with the usual angle relations to find the exact value.

One can also use trigonometric techniques, like sine rule, to solve this problem.

B4. We want to find a nonzero polynomial $p(x)$ with integer coefficients having the following property.

$$\text{Letting} q(x) := \frac{p(x)}{x(1-x)}, \quad q(x) = q\left(\frac{1}{1-x}\right).$$

Solution: First derive the following relationship

$$p(1-x) = -x^3 p\left(\frac{1}{x}\right).$$

This implies that the degree of the polynomial is at most 3. An easy check shows that linear polynomials can't satisfy this relation. Substituting $p(x) = ax^2 + bx + c$ in the above relation gives us that

$$p(x) = ax(x-1).$$

Let the cubic be of the form

$$p(x) = ax^3 + bx^2 + cx + d.$$

Then any cubic whose coefficients satisfy the following is a candidate.

$$a + d = 0, c - b - 3a = 0, 3b + c + 3a = 0, b + c + d = 0.$$

B5. [**14 points**] Let $\mathbb{R}_+$ denote the set of positive real numbers. A bijection $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called golden if $f'(x) = f^{-1}(x)$ for every $x \in \mathbb{R}_+$.

1. Find all golden functions (if any) of the form $f(x) = ax^b$. Find all golden functions (if any) of the form $f(x) = ab^x$.

2. Show that there is no bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f^{-1}(x)$.

Solution:

1. When $f(x) = ax^b$ equate the derivative with the inverse to get $b^2 - b - 1 = 0$. Whose roots are the golden ration, ϕ, and its negative inverse, $1 - \phi$. The answer is

$$f(x) = (x(\phi - 1))^{\phi} \quad \text{OR} \quad f(x) = (\frac{x}{\phi - 1})^{1-\phi}.$$

   On the other hand $f(x) = ab^x$ is not a candidate for the golden function since the derivative is an exponential and the inverse is a logarithmic function.

2. If f is a bijection defined on the entire real line then the derivative doesn't change its sign. However, the inverse does change sign. Hence it is impossible that the derivative is the inverse of such a function.

B6. [**14 points**] Suppose $n > 1$ is a natural number which is not congruent to 3 modulo 4. Prove that there exist $1 \leq i < j \leq n$ such that the following is a perfect square

$$\frac{1!2!\cdots n!}{i!j!}.$$

Solution: The important observaion needed is that $k!(k-1)! = k((k-1)!)^2$. Case 1: $n = 4k$ for some k. In that case the numerator simplifies as follows:

$$\begin{aligned} 1!2!\cdots(4k)! &= 4k((4k-1)!)^2\cdots 2\cdot 1 \\ &= (4k)(4k-2)\cdots 2\cdot M^2 \\ &= 2^{2k}\cdot 2k\cdot(2k-1)\cdots 1\cdot M^2 \\ &= (2k)!\cdot N^2. \end{aligned}$$

Here M^2 is the product of the terms of the form $((4k-1)!)^2$ and N^2 includes the factor $(2^k)^2$. Hence one can choose $i = 1$ and $j = 2k$. The other two remaining cases of n can be dealt in a similar fashion.