2020 Entrance Examination for BSc Programmes at CMI

Part A. Write your final answers on page 3 of the answer booklet.

Part A is worth a total of 40 (= 4 × 10) points. Points will be given based only on clearly legible final answers filled in the correct place on page 3 of the answer booklet. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a number (rational/real/complex) or, where appropriate, ∞ or $-\infty$. If a desired answer "does not exist" or is "not possible to decide", state so. Write integer answers in the usual decimal form. Write non-integer rationals as ratios of two integers.

A1. Each student in a small school has to be a member of at least one of THREE school clubs. It is known that each club has 35 members. It is not known how many students are members of two of the three clubs, but it is known that exactly 10 students are members of all three clubs. What is the largest possible total number of students in the school? What is the smallest possible total number of students in the school?

A2. Let P be the plane containing the vectors $(6, 6, 9)$ and $(7, 8, 10)$. Find a unit vector that is perpendicular to $(2, -3, 4)$ and that lies in the plane P. (Note: all vectors are considered as line segments starting at the origin $(0, 0, 0)$. In particular the origin lies in the plane P.)

A3. Calculate the following two definite integrals. It may be useful to first sketch the graph.

$$\int_1^{e^2} \ln|x|\,dx \qquad\qquad \int_{-1}^{1} \frac{\ln|x|}{|x|}\,dx$$

A4. A fair die is thrown 100 times in succession. Find probabilities of the following events.

(i) 4 is the outcome of one or more of the first three throws.

(ii) Exactly 2 of the last 4 throws give an outcome divisible by 3 (i.e., outcome 3 or 6).

A5. Write your answers to each question below as a series of three letters Y (for Yes) or N (for No). Leave space between the group of three letters answering (i), the answers to (ii) and the answers to (iii). Consider the graphs of functions

$$f(x) = \frac{x^3}{x^2 - x} \qquad\qquad g(x) = \frac{x^2 - x}{x^3} \qquad\qquad h(x) = \frac{x^3 - x}{x^3 + x}.$$

(i) Does f have a horizontal asymptote? A vertical asymptote? A removable discontinuity?

(ii) Does g have a horizontal asymptote? A vertical asymptote? A removable discontinuity?

(ii) Does h have a horizontal asymptote? A vertical asymptote? A removable discontinuity?

A6. Recall the function $\arctan(x)$, also denoted as $\tan^{-1}(x)$. Complete the sentence:

$$\arctan(20202019) + \arctan(20202021) \qquad \_\_\_\_\_ \qquad 2\arctan(20202020),$$

because in the relevant region, the graph of $y = \arctan(x)$ _____ .

Fill in the first blank with one of the following: is less than / is equal to / is greater than. Fill in the second blank with a single correct reason consisting of one of the following phrases: is bounded / is continuous / has positive first derivative / has negative first derivative / has positive second derivative / has negative second derivative / has an inflection point.

A7. The polynomial $p(x) = 10x^{400} + ax^{399} + bx^{398} + 3x + 15$, where a, b are real constants, is given to be divisible by $x^2 - 1$.

(i) If you can, find the values of a and b. Write your answers as $a =$ _____, $b =$ _____. If it is not possible to decide, state so.

(ii) If you can, find the sum of *reciprocals* of all 400 (complex) roots of $p(x)$. Write your answer as sum $=$ _____. If it is not possible to decide, state so.

A8. For a positive integer n, let $D(n) =$ number of positive integer divisors of n. For example, $D(6) = 4$ because 6 has four divisors, namely 1, 2, 3 and 6. Find the number of $n \leq 60$ such that $D(n) = 6$.

A9. Notice that the quadratic polynomial $p(x) = 1 + x + \frac{1}{2}x(x-1)$ satisfies $p(j) = 2^j$ for $j = 0, 1$ and 2. A polynomial $q(x)$ of degree 7 satisfies $q(j) = 2^j$ for $j = 0, 1, 2, 3, 4, 5, 6, 7$. Find the value of $q(10)$.

A10. Note that $25 \times 16 - 19 \times 21 = 1$. Using this or otherwise, find positive integers a, b and c, all $\leq 475 = 25 \times 19$, such that

- a is 1 mod 19 and 0 mod 25,
- b is 0 mod 19 and 1 mod 25, and
- c is 4 mod 19 and 10 mod 25.

(Recall the mod notation: since 13 divided by 5 gives remainder 3, we say 13 is 3 mod 5.)

Part B. Write solutions on pages 4 to 15 in the answer booklet.

Part B is worth a total of 60 points. Clearly explain your entire reasoning. No credit will be given without reasoning. Partial solutions may get partial credit.

B1. [7 points] Suppose A, B, C, D are points on a circle such that AC and BD are diameters of that circle. Suppose AB = 12 and BC = 5. Let P be a point on the arc of the circle from A to B (the arc that does not contain points C and D). Let the distances of P from A, B, C and D be a, b, c and d respectively. Find the values of $\frac{a+b}{c+d}$ and $\frac{a-b}{d-c}$. You may assume $d \neq c$ so the second ratio makes sense.

B2. [7 points] Let $z = e^{(\frac{2\pi i}{n})}$. Here $n \geq 2$ is a positive integer, $i^2 = -1$ and the real number $\frac{2\pi}{n}$ can also be considered as an angle in radians.

(i) Show that $\sum_{k=0}^{n-1} z^k = 0$. (ii) Show that $\sum_{k=0}^{8} \cos(40k+1)^\circ = 0$, i.e.,

$$\cos(1^\circ) + \cos(41^\circ) + \cos(81^\circ) + \cos(121^\circ) + \cdots + \cos(241^\circ) + \cos(281^\circ) + \cos(321^\circ) = 0.$$

B3. [10 points] A spider starts at the origin and runs in the first quadrant along the graph of $y = x^3$ at the constant speed of 10 unit/second. The speed is measured along the length of the curve $y = x^3$. The formula for the curve length along the graph of $y = f(x)$ from $x = a$ to $x = b$ is $\ell = \int_a^b \sqrt{1 + f'(x)^2}\, dx$. As the spider runs, it spins out a thread that is always maintained in a straight line connecting the spider with the origin. What is the rate in unit/second at which the thread is elongating when the spider is at $(\frac{1}{2}, \frac{1}{8})$?

You should use the following names for variables. At any given time t, the spider is at the point (u, u^3), the length of the thread joining it to the origin in a straight line is s and the curve length along $y = x^3$ from the origin till (u, u^3) is ℓ. You are asked to find $\frac{ds}{dt}$ when $u = \frac{1}{2}$. (Do not try to evaluate the integral for ℓ: it is unnecessary and any attempt to do so will not get any credit because a closed formula in terms of basic functions does not exist.)

B4. [12 points] Throughout this problem we are interested in real valued functions f satisfying two conditions: at each x in its domain, f is continuous and $f(x^2) = f(x)^2$. Prove the following independent statements about such functions. The hints below may be useful.

(i) There is a unique such function f with domain [0,1] and $f(0) \neq 0$.

(ii) If the domain of such f is $(0, \infty)$, then ($f(x) = 0$ for every x) OR ($f(x) \neq 0$ for every x).

(iii) There are infinitely many such f with domain $(0, \infty)$ such that $\int_0^\infty f(x)\, dx < 1$.

Hints: (1) Suppose a number a and a sequence x_n are in the domain of a continuous function f and x_n converges to a. Then $f(x_n)$ must converge to $f(a)$. For example $f(0.5^n) \to f(0)$ and $f(2^{\frac{1}{n}}) \to f(1)$ if all the mentioned points are in the domain of f. In parts (i) and (ii) suitable sequences may be useful. (2) Notice that $f(x) = x^r$ satisfies $f(x^2) = f(x)^2$.

B5. [12 points] Consider polynomials $p(x)$ with the following property, called (†).

(†) If r is a root of $p(x)$, then $r^2 - 4$ is also a root of $p(x)$.

(i) We want to find every quadratic polynomial of the form $p(x) = x^2 + bx + c$ such that $p(x)$ has two distinct roots, has integer coefficients and has property (†). Prove that there are exactly two such polynomials and list them in the provided space on a later page.

(ii) It is also true that there are exactly two cubic polynomials of the form $p(x) = x^3 + ax^2 + bx + c$ with the property (†) such that $p(x)$ shares no root with the polynomials you found in part (i). Explain fully how you will prove this along with the method to find the polynomials, but do not try to explicitly find the polynomials.

B6. [12 points] For sets S and T, a relation from S to T is just a subset R of $S \times T$. If (x, y) is in R, we say that x is related to y. Answer the following. Part (i) is independent of (ii) and (iii).

(i) A relation R from S to S is called *antisymmetric* if it satisfies the following condition: if (a, b) is in R, then (b, a) must NOT be in R. For $S = \{1, 2, \dots, k\}$, how many antisymmetric relations are there from S to S?

(ii) Write a recurrence equation for $f(k, n) =$ the number of *non-crossing* relations from $\{1, 2, \dots k\}$ to $\{1, 2, \dots n\}$ that have *no isolated elements* in either set. (See below for the definitions of the two *underlined* terms and their visual meaning. Drawing pictures may be useful.) Your recurrence should have only a fixed number of terms on the RHS.

(iii) Using your recurrence in (ii) or otherwise, find a formula for $f(3, n)$.

Definition 1: We say that a relation from S to T has *no isolated elements* if each s in S is related to some t in T and if for each t in T, some s in S is related to t.

Definition 2: We say that a relation R from $\{1, 2, \dots k\}$ to $\{1, 2, \dots n\}$ is *non-crossing* if the following never happens: (i, x) and (j, y) are both in R with $i < j$ but $x > y$.

Visual meaning: one can visualise a relation R very similarly to a function. List 1 to k as dots arranged vertically in increasing order on the left and similarly list 1 to n on the right. For each (s, t) in R, draw a straight line segment from s on the left to t on the right. In the situation one wants to avoid for non-crossing relations, the segments connecting i with x and j with y would cross. Having no isolated elements also has an obvious visual meaning.