CHENNAI MATHEMATICAL INSTITUTE

Undergraduate Programme in Mathematics and Computer Science/Physics
Common Entrance Examination
15^{th} May 2019

Enter your *Admit Card Number*: U – ☐☐☐ – ☐☐☐☐

IMPORTANT INSTRUCTIONS!

- **Ensure that this booklet has all 18 printed sides**: this cover page and 17 numbered pages. Pages numbered 1 to 3 contain *ten* questions in part A plus the space to answer part A on page 3. Page numbered 4 contains *six* questions in part B. Pages 6 to 17 contain answer sheets for each question in part B. For rough work use the colored blank pages at the end.
- **Time allowed is 3 hours. Total points: 100** = 40 for part A + 60 for part B.
- **Part A will be used for screening.** Part B will be graded only if you score a certain minimum in part A. However your scores in both parts will be used while making the final decision. Specific instructions for each part are given below.
- **Advice:** Please ensure that you have about 2 hours left for part B.

For office use only

$\times$	Points	Remarks
Part A		
Part B		
Total		

	Points	Remarks
B1		
B2		
B3		
B4		
B5		
B6		
Total		

2019 Entrance Examination for the BSc Programmes at CMI

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of (4 × 10 = 40) points. Points will be given based only on clearly legible final answers filled in the correct place on page 3. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a number (rational/ real/ complex) or, where appropriate, one of the phrases "infinite"/"does not exist"/"not possible to decide". Write integer answers in the usual decimal form. Write non-integer rationals as ratios of two integers.

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1. For a natural number m, define $\Phi_1(m)$ to be the number of divisors of m and for $k \geq 2$ define $\Phi_k(m) := \Phi_1(\Phi_{k-1}(m))$. For example, $\Phi_2(12) = \Phi_1(6) = 4$. Find the minimum k such that
$$\Phi_k(2019^{2019}) = 2.$$
2. Let f be a real valued continuous function defined on $\mathbb{R}$ satisfying
$$f'(\tan^2\theta) = \cos 2\theta + \tan\theta \sin 2\theta, \text{ for all real numbers } \theta.$$
If $f'(0) = -\cos\frac{\pi}{12}$ then find $f(1)$.
3. You have a piece of land close to a river, running straight. You are required to cut off a rectangular portion of the land, with the river forming one of the sides of the rectangle so, your fence will have three sides to it. You only have 60 meters of fencing. The maximum area that you can enclose is ....
4. The sum
$$S = 1 + 111 + 11111 + \cdots + \underbrace{11\cdots1}_{2k+1}$$
is equal to ....
5. You are given an 8×8 chessboard. If two distinct squares are chosen uniformly at random find the probability that two rooks placed on these squares attack each other. Recall that a rook can move either horizontally or vertically, in a straight line.
6. For how many natural numbers n is $n^6 + n^4 + 1$ a square of a natural number?
7. A broken calculator has all its 10 digit keys and two operation keys intact. Let us call these operation keys A and B. When the calculator displays a number n pressing A changes the display to $n+1$. When the calculator displays a number n pressing B changes the display to $2n$. For example, if the number 3 is displayed then the key strokes ABBA changes the display in the following steps $3 \to 4 \to 8 \to 16 \to 17$.

   If 1 is on the display what is the least number of key strokes needed to get 260 on the display?

8. Let $\pi = \pi_1\pi_2 \ldots\ldots \pi_n$ be a permutation of the numbers $1, 2, 3, \ldots, n$. We say π has its first ascent at position $k < n$ if $\pi_1 > \pi_2 \ldots > \pi_k$ and $\pi_k < \pi_{k+1}$. If $\pi_1 > \pi_2 > \ldots > \pi_{n-1} > \pi_n$ we say π has its first ascent in position n. For example when $n = 4$ the permutation 2134 of has its first ascent at position 2.

   The number of permutations which have their first ascent at position k is ....

   For questions 9 and 10 below, some statements are given. For each statement, state if it is true or false. Write your answer to each question as a sequence of three/ four letters (T for True and F for False) in correct order.

9. Consider $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(a, b) := \lim_{n\to\infty} \frac{1}{n} \log_e[e^{na} + e^{nb}].$$

   (a) f is not onto i.e. the range of f is not all of $\mathbb{R}$.

   (b) For every a the function $x \mapsto f(a, x)$ is continuous everywhere.

   (c) For every b the function $x \mapsto f(x, b)$ is differentiable everywhere.

   (d) We have $f(0, x) = x$ for all $x \geq 0$.

10. Let $f : \mathbb{R} \to \mathbb{R}$.

   (a) There is no continuous function f for which $\int_0^1 f(x)(1 - f(x))dx < \frac{1}{4}$.

   (b) There is only one continuous function f for which $\int_0^1 f(x)(1 - f(x))dx = \frac{1}{4}$.

   (c) There are infinitely many continuous functions f for which $\int_0^1 f(x)(1-f(x))dx = \frac{1}{4}$.

Answers to part A

This is the only place that will be seen for grading part A. So carefully and clearly write the answers to each question on the designated line below. Write only the final answers, do not show any intermediate work. Illegible/unclear answers will not be considered. $\times$

A1. ____

A2. ____

A3. ____

A4. ____

A5. ____

A6. ____

A7. ____

A8. ____

A9. ____

A10. ____

Part B. Write complete solutions for these questions from page 6 onwards.

Part B is worth a total of 60 marks. Solve these questions in the space provided for each question from page 6. You may solve only part of a question and get partial credit. **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. For a natural number n denote by $\mathrm{Map}(n)$ the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. For $f, g \in \mathrm{Map}(n)$, $f \circ g$ denotes the function in $\mathrm{Map}(n)$ that sends x to $f(g(x))$. [10 marks]

   (a) Let $f \in \mathrm{Map}(n)$. If for all $x \in \{1, \ldots, n\}$ $f(x) \neq x$, show that $f \circ f \neq f$.

   (b) Count the number of functions $f \in \mathrm{Map}(n)$ such that $f \circ f = f$.

2. (a) Count the number of roots w of the equation $z^{2019} - 1 = 0$ over complex numbers that satisfy $|w + 1| \geq \sqrt{2 + \sqrt{2}}$. [5 marks]

   (b) Find all real numbers x that satisfy following equation: [5 marks]

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

3. Evaluate $\int_0^\infty (1 + x^2)^{-(m+1)} dx$, where m is a natural number. [10 marks]

4. Let $ABCD$ be a parallelogram. Let O be a point in its interior such that $\angle AOB + \angle DOC = 180°$. Show that $\angle ODC = \angle OBC$. [10 marks]

5. Three positive real numbers x, y, z satisfy

$$\begin{aligned} x^2 + y^2 &= 3^2 \\ y^2 + yz + z^2 &= 4^2 \\ x^2 + \sqrt{3}xz + z^2 &= 5^2. \end{aligned}$$

   Find the value of $2xy + xz + \sqrt{3}yz$. [10 marks]

6. (a) Compute $\dfrac{d}{dx}\left[\displaystyle\int_0^{e^x} \log(t)\, \cos^4(t)\, dt\right]$. [4 marks]

   (b) For $x > 0$ define $F(x) = \int_1^x t \log(t) dt$. [6 marks]

      i. Determine the open interval(s) (if any) where $F(x)$ is decreasing and the open interval(s) (if any) where $F(x)$ is increasing.

      ii. Determine all the local minima of $F(x)$ (if any) and the local maxima of $F(x)$ (if any) .

Write answers to part B from the next page.
If you need extra space for any problem, continue on one of the colored blank pages at the end and write a note to that effect.

Answers to part B

1. For a natural number n denote by $\mathrm{Map}(n)$ the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. For $f, g \in \mathrm{Map}(n)$, $f \circ g$ denotes the function in $\mathrm{Map}(n)$ that sends x to $f(g(x))$. [10 marks]

   (a) Let $f \in \mathrm{Map}(n)$. If for all $x \in \{1, \ldots, n\}$ $f(x) \neq x$, show that $f \circ f \neq f$.

   (b) Count the number of functions $f \in \mathrm{Map}(n)$ such that $f \circ f = f$.

2. (a) Count the number of roots w of the equation $z^{2019} - 1 = 0$ over complex numbers that satisfy $|w + 1| \geq \sqrt{2 + \sqrt{2}}$. [5 marks]

(b) Find all real numbers x that satisfy following equation: [5 marks]

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

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4. Let $ABCD$ be a parallelogram. Let O be a point in its interior such that

$$\angle AOB + \angle DOC = 180^\circ.$$

Show that $\angle ODC = \angle OBC$. [10 marks]

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ii. Determine all the local minima of $F(x)$ (if any) and the local maxima of $F(x)$ (if any) .