2019 Entrance Examination for the national undergraduate programmes at CMI

Solutions to Part **A**

1. For a natural number m, define $\Phi_1(m)$ to be the number of divisors of m and for $k \geq 2$ define $\Phi_k(m) := \Phi_1(\Phi_{k-1}(m))$. For example, $\Phi_2(12) = \Phi_1(6) = 4$. Find the minimum k such that
$$\Phi_k(2019^{2019}) = 2.$$
Answer: 6

2. Let f be a real valued continuous function defined on $\mathbb{R}$ satisfying
$$f'(\tan^2\theta) = \cos 2\theta + \tan\theta \sin 2\theta, \text{ for all real numbers } \theta.$$
If $f(0) = -\cos\frac{\pi}{12}$ then find $f(1)$.
Note there was a typo in the exam; it was printed $f'(0)$ instead of $f(0)$. Answer: Put $y = tan^2\theta$. Then we have
$$f'(y) = 1$$
Hence the answer is $1 - \cos\frac{\pi}{12}$.

3. You have a piece of land close to a river, running straight. You are required to cut off a rectangular portion of the land, with the river forming one of the sides of the rectangle so, your fence will have three sides to it. You only have 60 meters of fencing. The maximum area that you can enclose is ....
Answer: 450 **square meters**.

4. The sum
$$S = 1 + 111 + 11111 + \cdots + \underbrace{11\cdots1}_{2k+1}$$
is equal to ....
Answer: $\dfrac{10^{2k+3} - 99k - 109}{99 \times 9}$.

5. You are given an 8×8 chessboard. If two distinct squares are chosen uniformly at random find the probability that two rooks placed on these squares attack each other. Recall that a rook can move either horizontally or vertically, in a straight line.
Answer: $\dfrac{2}{9}$.

6. For how many natural numbers n is $n^6 + n^4 + 1$ a square of a natural number?
Answer: 1 ($n = 2$ **is the only solution**.

7. A broken calculator has all its 10 digit keys and two operation keys intact. Let us call these operation keys A and B. When the calculator displays a number n pressing A changes the display to $n+1$. When the calculator displays a number n pressing B changes the display to $2n$. For example, if the number 3 is displayed then the key strokes ABBA changes the display in the following steps $3 \to 4 \to 8 \to 16 \to 17$.

   If 1 is on the display what is the least number of key strokes needed to get 260 on the display?

   Answer: 9, **there are exactly two sequences, for example**, $BBBBBBABB$.

8. Let $\pi = \pi_1\pi_2 \ldots\ldots \pi_n$ be a permutation of the numbers $1, 2, 3, \ldots, n$. We say π has its first ascent at position $k < n$ if $\pi_1 > \pi_2 \ldots > \pi_k$ and $\pi_k < \pi_{k+1}$. If $\pi_1 > \pi_2 > \ldots > \pi_{n-1} > \pi_n$ we say π has its first ascent in position n. For example when $n = 4$ the permutation 2134 of has its first ascent at position 2.

   The number of permutations which have their first ascent at position k is ....

   Answer: $\binom{n}{k}(n-k)! - \binom{n}{k+1}(n-k-1)!$.

   For questions 9 and 10 below, some statements are given. For each statement, state if it is true or false. Write your answer to each question as a sequence of three/ four letters (T for True and F for False) in correct order.

9. Consider $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(a,b) := \lim_{n\to\infty} \frac{1}{n} \log_e[e^{na} + e^{nb}].$$

   (a) f is not onto i.e. the range of f is not all of $\mathbb{R}$.

   (b) For every a the function $x \mapsto f(a,x)$ is continuous everywhere.

   (c) For every b the function $x \mapsto f(x,b)$ is differentiable everywhere.

   (d) We have $f(0,x) = x$ for all $x \geq 0$.

   Answer: **FTFT**

10. Let $f\colon \mathbb{R} \to \mathbb{R}$.

   (a) There is no continuous function f for which $\int_0^1 f(x)(1-f(x))dx < \frac{1}{4}$.

   (b) There is only one continuous function f for which $\int_0^1 f(x)(1-f(x))dx = \frac{1}{4}$.

   (c) There are infinitely many continuous functions f for which $\int_0^1 f(x)(1-f(x))dx = \frac{1}{4}$.

   Answer: **FFT**

Solutions to Part **B**

1. For a natural number n denote by Map(n) the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. For $f, g \in$ Map(n), $f \circ g$ denotes the function in Map(n) that sends x to $f(g(x))$. [10 marks]

(a) Let $f \in \text{Map}(n)$. If for all $x \in \{1, \ldots, n\}$ $f(x) \neq x$, show that $f \circ f \neq f$.
Answer: Suppose $f(f(x)) = f(x)$. Set $y = f(x)$. Then we have $f(y) = y$, a contraction.

(b) Count the number of functions $f \in \text{Map}(n)$ such that $f \circ f = f$.
Answer: Note that from the above part it follows that each x has to map to a fixed point of f (i.e., a y such that $f(y) = y$. So in order to count the number of such functions we first need to decide the number of fixed points. The number functions that have exactly k fixed points is $\binom{n}{k}k^{n-k}$. In order to get the total number sum the previous quantity over $1 \leq k \leq n$.

2. (a) Count the number of roots w of the equation $z^{2019} - 1 = 0$ over complex numbers that satisfy $|w+1| \geq \sqrt{2+\sqrt{2}}$. [5 marks]
Answer: Such roots can be expressed as follows

$$w = \frac{\cos(2\pi k)}{2019} + i\frac{\sin(2\pi k)}{2019} \quad \textbf{for } k = 0, \pm 1, \ldots, \pm 1009.$$

Therefore,

$$|w+1|^2 = 2 + 2\frac{\cos(2\pi k)}{2019}.$$

Hence we want to identify k such that

$$\frac{\cos(2\pi k)}{2019} \geq \frac{1}{\sqrt{2}}.$$

Which is equivalent to

$$|\frac{2\pi k}{2019}| \leq \frac{\pi}{4}$$
$$\textbf{i.e., } |k| \leq 252.$$

So there are 505 solutions satisfying the given inequality.

(b) Find all real numbers x that satisfy following equation: [5 marks]

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

Answer: Put $a = 2^x$ and $b = 3^x$. This reduces the given equation to the following quadratic -

$$6a^2 - 13ab + 6b^2 = 0.$$

Solving the above equation and re-substituting we get $x = \pm 1$.

3. Evaluate $\int_0^\infty (1+x^2)^{-(m+1)}dx$, where m is a natural number. [10 marks]
Answer: There are various ways to solve this. One can start with the subsitution $x = \tan u$, that changes the integral to

$$I = \frac{1}{4}\int_0^{2\pi} (\cos(u))^{2m}\, du.$$

Using integration by reduction technique the final answer is

$$\frac{2\pi}{4} \cdot \frac{(2m)!}{2^{2m}(m!)^2}.$$

4. Let $ABCD$ be a parallelogram. Let O be a point in its interior such that $\angle AOB + \angle DOC = 180°$. Show that $\angle ODC = \angle OBC$. [10 marks]
   Answer: Note that there exists an external point P such that AP is parallel to DO, BP is parallel to CO and OP is parallel to BC. Now $AOBP$ is a cyclic quadrilateral. Rest is a straightforward calculation involving angles.

5. Three positive real numbers x, y, z satisfy

$$\begin{aligned} x^2 + y^2 &= 3^2 \\ y^2 + yz + z^2 &= 4^2 \\ x^2 + \sqrt{3}xz + z^2 &= 5^2. \end{aligned}$$

   Find the value of $2xy + xz + \sqrt{3}yz$. [10 marks]
   Answer: Consider the right angled triangle ABC with sides $3, 4, 5$ and an interior point O such that $AO = x, \angle AOB = 90$ and $CO = z, \angle COA = 150$ and $BO = y, \angle BOC = 120$. Then the three given equations are in fact cosine rule for each of the triangle prescribed above. For example, in ΔBOC we have

$$\begin{aligned} 4^2 &= y^2 + z^2 - 2yz\cos(120) \\ &= y^2 + z^2 + yz. \end{aligned}$$

   The area of δABC (which is 6) calculated using the sine formula (for each of the smaller triangle) gives us

$$6 = \frac{1}{2}xy + \frac{1}{2}yz\sin 60 + \frac{1}{2}\sin 30$$

   So the answer is 24.

6. (a) Compute $\dfrac{d}{dx}\left[\displaystyle\int_0^{e^x} \log(t)\, \cos^4(t)\, dt\right]$. [4 marks]

   (b) For $x > 0$ define $F(x) = \int_1^x t\log(t)dt$. [6 marks]
   Answer: This is fairly straightforward: substitute e^y for t and use the Leibniz rule for the differntial under an integral sign to get the answer

$$(e^x)(\log(e^x))\cos^4(e^x).$$

   i. Determine the open interval(s) (if any) where $F(x)$ is decreasing and the open interval(s) (if any) where $F(x)$ is increasing.
   ii. Determine all the local minima of $F(x)$ (if any) and the local maxima of $F(x)$ (if any) .

Answer:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_1^x t \, \log t \, \mathrm{d}t \\ &= x \log x. \end{aligned}$$

Therefore $F'(1) = 0$. **Moreover,** $F''(x) = 1 + \log x$. **Hence one concludes that** F **is decreasing on** $(0, 1)$, **increasing on** $(1, \infty)$ **and has a local minima at** $x = 1$.