Solutions to 2014 Entrance Examination for BSc Programmes at CMI

A1. Let α, β and c be positive numbers less than 1, with c rational and α, β irrational.

(A) The number $\alpha + \beta$ must be irrational.
(B) The infinite sum $\sum_{i=0}^{\infty} \alpha c^i = \alpha + \alpha c + \alpha c^2 + \cdots$ must be irrational.
(C) The value of the integral $\int_0^{\pi} (\beta \cos x + c)\, dx$ must be irrational.

Answer: FTT

A2. Consider the intergal $I = \int_1^{\infty} e^{ax^2+bx+c}\, dx$, where a, b, c are constants. Some combinations of values for these constants are given below and you have to decide in each case whether the integral I converges.

(A) I converges for $a = -1 \quad b = 10 \quad c = 100$.
(B) I converges for $a = 1 \quad b = -10 \quad c = -100$.
(C) I converges for $a = 0 \quad b = -1 \quad c = 100$.
(D) I converges for $a = 0 \quad b = 0 \quad c = -100$.

Answer: TFTF

A3. Given a real number x, define $g(x) = x^2 e^x$ if $x \geq 0$ and $g(x) = xe^{-x}$ if $x < 0$.

(A) The function g is continuous everywhere.
(B) The function g is differentiable everywhere.
(C) The function g is one-to-one.
(D) The range of g is the set of all real numbers.

Answer: TFTT

A4. Find the slope of a line L that satisfies both of the following properties: (i) L is tangent to the graph of $y = x^3$. (ii) L passes through the point (0,2000).

Answer: 300 (Note: if the point in (ii) is taken to be (0,200), then the answer is $30(10)^{\frac{1}{3}}$.)

A5. A regular 100-sided polygon is inscribed in a circle. Suppose three of the 100 vertices are chosen at random, all such combinations being equally likely. Find the probability that the three chosen points form vertices of a right angled triangle.

Answer: $\frac{1}{33}$

A6. What is the smallest positive integer n for which $\frac{50!}{24^n}$ is *not* an integer?

Answer: 16

A7. Let $f(x) = (x-a)(x-b)^3(x-c)^5(x-d)^7$, where a, b, c, d are real numbers with $a < b < c < d$. Thus $f(x)$ has 16 real roots counting multiplicities and among them 4 are distinct from each other. Consider $f'(x)$, i.e. the derivative of $f(x)$. Find the following, if you can: (i) the number of real roots of $f'(x)$, counting multiplicities, (ii) the number of *distinct* real roots of $f'(x)$.

Answers: 15, 6

A8. Let $f(x) = 7x^{32} + 5x^{22} + 3x^{12} + x^2$. (i) Find the remainder when $f(x)$ is divided by $x^2 + 1$. (ii) Find the remainder when $xf(x)$ is divided by $x^2 + 1$. In each case your answer should be a polynomial of the form $ax + b$, where a and b are constants.

Answers: $0x + 4, 4x + 0$

A9. Let $\theta_1, \theta_2, \ldots, \theta_{13}$ be *real* numbers and let A be the average of the complex numbers $e^{i\theta_1}, e^{i\theta_2} \ldots, e^{i\theta_{13}}$, where $i = \sqrt{-1}$. As the values of θ's vary over all 13-tuples of real numbers, find (i) the maximum value attained by $|A|$, (ii) the minimum value attained by $|A|$.

Answers: 1, 0

A10. In each of the following independent situations we want to construct a triangle ABC satisfying the given conditions. In each case state state how many such triangles ABC exist up to congruence.

(i) $AB = 30 \quad BC = 95 \quad AC = 55$

(ii) $\angle A = 30° \quad \angle B = 95° \quad \angle C = 55°$

(iii) $\angle A = 30° \quad \angle B = 95° \quad BC = 55$

(iv) $\angle A = 30° \quad AB = 95 \quad BC = 55$

Answers: 0, infinite, 1, 2

A11. Let A_n = the area of a regular n-sided polygon inscribed in a circle of radius 1 (i.e., vertices of this regular n-sided polygon lie on a circle of radius 1). (i) Find A_{12}. (ii) Find $\lfloor A_{2014} \rfloor$, i.e., the greatest integer $\leq A_{2014}$.

Answers: 3, 3

A12. The total length of all 12 sides of a rectangular box is 60. (i) Write the possible values of the volume of the box. Your answer should be an interval. Now suppose in addition that the surface area of the box is given to be 56. Find, if you can, (ii) the length of the longest diagonal of the box (iii) the volume of the box.

Answers: (0, 125], 13, not possible to decide

B1. Find the area of the region in the XY plane consisting of all points in the set

$$\{(x, y)|x^2 + y^2 \leq 144 \text{ and } \sin(2x + 3y) \leq 0\}.$$

Answer: The area of the circular region $S = \{(x, y)|x^2 + y^2 \leq 144\}$ is 144π. The condition $\sin(2x + 3y) \leq 0$ is equivalent to $2x + 3y$ being in one of the intervals $[k\pi, (k + 1)\pi]$, where k is an odd integer. The key point is that due to the symmetry of the circle S about any diameter, in particular the diameter $2x + 3y = 0$, the strip inside S lying between the lines $2x + 3y = k\pi$ and $2x + 3y = (k + 1)\pi$ is the mirror image of strip lying between the lines $2x + 3y = -k\pi$ and $2x + 3y = -(k + 1)\pi$. For each integer k, precisely one of these two equal strips is included in the desired area. Thus the desired area is half that of S, i.e., 72π.

B2. Let x be a real number such that $x^{2014} - x^{2004}$ and $x^{2009} - x^{2004}$ are both integers. Show that x is an integer. (Hint: it may be useful to first prove that x is rational.)

Answer: Here is one of several possible ways. $x^{2014} - x^{2009} = x^{2009}(x^5 - 1)$ and $x^{2004}(x^5 - 1)$ are integers, which we may assume to be nonzero (else $x = 0$ or 1 and we are done). Dividing, we get that x^5 is rational. Now dividing the integer $x^{2004}(x^5 - 1)$ by the rational number $x^5 - 1$, we see that x^{2004} is rational. Since 2004 and 5 are coprime, x is rational as well. (E.g., x^5 is rational, so $(x^5)^{401} = x^{2005}$ is rational. Now divide by the rational number x^{2004}.)

Let $x = \frac{a}{b}$ with a, b coprime integers. Consider the integer $\frac{a^{2009}}{b^{2009}} - \frac{a^{2004}}{b^{2004}} = \frac{a^{2009} - b^5 a^{2004}}{b^{2009}}$. If a prime p divides the denominator, it must divide the numerator as well. Now $p|b$, so $p|b^5 a^{2004}$, so $p|a^{2009}$ and finally $p|a$, a contradiction. Thus $b = 1$, i.e., x is an integer.

B3. (i) How many functions are there from the set $\{1, \dots, k\}$ to the set $\{1, \dots, n\}$?

(ii) Let P_k denote the set of all subsets of $\{1, \dots, k\}$. Find a formula for the number of functions f from P_k to $\{1, \dots, n\}$ such that $f(A \cup B) =$ the larger of the two integers $f(A)$ and $f(B)$. Your answer need not be a closed formula but it should be simple enough to use for given values of n and k, e.g., to see that for $k = 3$ and $n = 4$ there are 100 such functions.

Example: When $k = 2$, the set P_2 contains 4 elements: the empty set $\phi, \{1\}, \{2\}$ and $\{1, 2\}$. The function f given by $\phi \to 2, \{1\} \to 3, \{2\} \to 4, \{1, 2\} \to 4$ satisfies the given condition. But the function g given by $\phi \to 2, \{1\} \to 3, \{2\} \to 4, \{1, 2\} \to 5$ does not, because $g(\{1\} \cup \{2\}) = g(\{1, 2\}) = 5 \neq$ the larger of $g(\{1\})$ and $g(\{2\}) = \max(3, 4) = 4$.

Answer: (i) As there are n choices each for the values of $f(1), \dots, f(k)$ and as all these choices are independent of each other, the number of functions is n^k.

(ii) Note that $f(A) = \max\ \{f(\{j\})|j \in A\}$, so the function f is completely decided by its values on the empty set ϕ and on the one element subsets $\{1\}, \{2\}, \dots, \{k\}$. If $f(\phi) = i$, then each of $f(\{1\}), f(\{2\}), \dots, f(\{k\})$ can be chosen to be any of the numbers $i, i + 1, \dots, n$. Thus there are k independent choices for each of which there are $n - i + 1$ options. So the number of desired functions for which $f(\phi) = i$ is $(n - i + 1)^k$. Now we sum over all values of $i = 1, 2, \dots, n$ to get the total number to be $1^k + 2^k + \cdots + n^k$. (When $k = 3$ and $n = 4$, we get $1^3 + 2^3 + 3^3 + 4^3 = 100$, as mentioned in the problem.)

B4. (i) Let f be continuous on $[-1,1]$ and differentiable at 0. For $x \neq 0$, define a function g by $g(x) = \frac{f(x)-f(0)}{x}$. Can $g(0)$ be defined so that the extended function g is continuous at 0?

(ii) For f as in part (i), show that the following limit exists.

$$\lim_{r \to 0^+} \left(\int_{-1}^{-r} \frac{f(x)}{x}\, dx + \int_r^1 \frac{f(x)}{x}\, dx \right)$$

(iii) Give an example showing that without the hypothesis of f being differentiable at 0, the conclusion in (ii) need not hold.

Answer: (i) Yes. We must define $g(0) = \lim_{x\to 0} g(x) = f'(0)$, which exists by hypothesis.

(ii) Consider $\int_{-1}^{-r} g(x)\, dx = \int_{-1}^{-r} \frac{f(x)}{x}\, dx - \int_{-1}^{-r} \frac{f(0)}{x}\, dx = \int_{-1}^{-r} \frac{f(x)}{x}\, dx - f(0)\ln r$. Similarly $\int_r^1 g(x)\, dx = \int_r^1 \frac{f(x)}{x}\, dx - \int_r^1 \frac{f(0)}{x}\, dx = \int_r^1 \frac{f(x)}{x}\, dx + f(0)\ln r$. (Or observe that since $\frac{1}{x}$ is an odd function, for $0 < a < b$, $\int_a^b \frac{1}{x}\, dx = -\int_{-b}^{-a} \frac{1}{x}\, dx$.) Thus the expression inside the given limit is equal to $\int_{-1}^{-r} g(x)\, dx + \int_r^1 g(x)\, dx$, as $\pm f(0)\ln r$ cancels out.

Applying the fundamental theorem of calculus to the continuous function g, we get an antiderivative G of g, where G is defined on $[-1,1]$ by $G(t) = \int_{-1}^t g(x)\, dx$. So the given limit $= \lim_{r\to 0^+}(\int_{-1}^{-r} g(x)\, dx + \int_r^1 g(x)\, dx) = \lim_{r\to 0^+}(G(-r) - G(-1) + G(1) - G(r)) = G(0) - 0 + G(1) - G(0) = G(1)$, where we have used the fundamental theorem to calculate the integrals and the fact that G, being differentiable, is also continuous.

(iii) Define $f(x) = \frac{1}{-\ln \frac{x}{2}}$ for $x \in (0,1]$, $f(x) = \frac{1}{\ln|\frac{x}{2}|}$ for $x \in [-1,0)$, and $f(0) = 0$. Verify that this works: f is continuous at 0 and so on $[-1,1]$. It is not differentiable at 0 as the relevant limit is $+\infty$. The two integrals in the desired limit are equal (because f is an odd function, so $\frac{f(x)}{x}$ is even) and each integral is $+\infty$ as it amounts to $\lim_{t\to 0^+} \ln|\ln t|$. Can you see how one might *think* of such f? E.g., check that choices like $|x|$ or even $x^{\frac{1}{3}}$ do not work. Compare the behaviour of these functions at $x = 0$ with that of chosen f. (Minor point: we used $\frac{x}{2}$ instead of x only to avoid trouble with dividing by $\ln|x|$ at endpoints $x = \pm 1$. We could have used $\frac{1}{\pm \ln|x|}$ if a smaller interval of definition is allowed, e.g., $x \in [-0.9, 0.9]$).

B5. (i) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial, where $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. Define the "discrete derivative of f", denoted Δf, to be the function given by $\Delta f(x) = f(x) - f(x-1)$. Show that Δf is also a polynomial and find its leading term.

(ii) For integers $n \geq 0$, define polynomials p_n of degree n as follows: $p_0(x) = 1$ and for $n > 0$, let $p_n(x) = \frac{1}{n!} x(x-1)(x-2)\cdots(x-n+1)$. So we have

$$p_0(x) = 1 \quad , \quad p_1(x) = x \quad , \quad p_2(x) = \frac{x(x-1)}{2} \quad , \quad p_3(x) = \frac{x(x-1)(x-2)}{3!} \quad \cdots$$

Show that for any polynomial f of degree n, there exist unique real numbers $b_0, b_1, \ldots, b_n$ such that $f(x) = \sum_{i=0}^n b_i p_i(x)$.

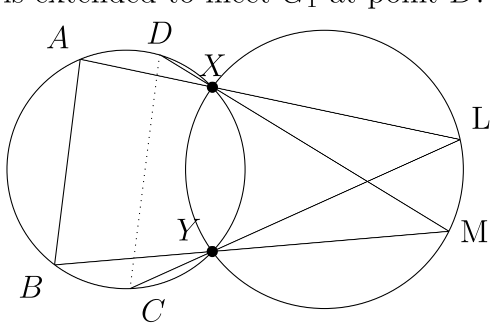
(iii) Now suppose that $f(x)$ is a polynomial such that for each integer m, $f(m)$ is also an integer. Using the above parts (or otherwise), show that for such f, the b_i obtained in part (ii) are integers.

Answer: (i) It is obvious that $f(x-1)$ is a polynomial in x (use binomial expansion of powers of $(x-1)$) and therefore so is $\Delta f(x) = f(x) - f(x-1)$, being the difference of polynomials. The point of the question is to find the leading term of $\Delta f(x)$. It is easy to see that after expanding the powers of $(x-1)$, the degree n terms of $f(x)$ and $f(x-1)$ cancel out, as does the degree $n-1$ term from $f(x)$ with the leading term of $a_{n-1}(x-1)^{n-1}$. The only remaining term of degree $n-1$ comes from $a_n(x-1)^n$. So $\Delta f(x) = na_nx^{n-1}+$ lower degree terms. Compare with the usual derivative.

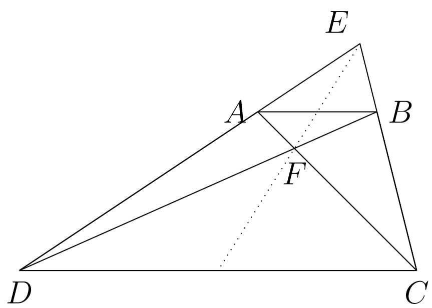
(ii) Induction on the degree of f. If $f(x) = a_0$ is constant, $b_0 = a_0$ works uniquely. Assuming the result for polynomials of degree $< n$, let f be of degree n, so $a_n \neq 0$. We are forced to take $b_n = n!a_n$ by comparing leading coefficients of $f(x)$ and $\sum_{i=0}^{n} b_ip_i(x)$. Now $f(x)-b_np_n(x)$ is a polynomial of degree $d < n$ and hence by induction equals $\sum_{i=0}^{d} b_ip_i(x)$ for unique $b_0, \ldots, b_d$. Therefore $f(x) = \sum_{i=0}^{n} b_ip_i(x)$, where $b_{d+1}, \ldots, b_{n-1}$ are all 0. To see uniqueness of b_i's, let $\sum_{i=0}^{n} b_ip_i(x) = \sum_{i=0}^{n} c_ip_i(x)$. Subtract all terms with $b_i = c_i$. If any terms are remaining, compare the leading coefficients on each side to get a contradiction.

(iii) Substitute $x = 0, 1, 2, \ldots$ one by one in the equation $f(x) = \sum_{i=0}^{n} b_ip_i(x)$ and solve for $b_0, b_1, b_2, \ldots$ successively. $x = 0$ gives $b_0 = f(0)$. Using $x = 1$ and 2 gives $b_1 = f(1) - b_0$, $b_2 = f(2) - b_0 - 2b_1$. In general, for all integers t, $p_i(t) = \binom{t}{i}$ is an integer. Further, $p_i(t) = 0$ if $0 \leq t < i$ and 1 if $t = i$. So $b_t = f(t) - \sum_{i=0}^{t-1} b_i\binom{t}{i}$, which is an integer by induction. (Note: We can also argue from the other end as follows. By repeated use of part (i), $\Delta^n f$, the n-th discrete derivative of f, is the constant $n!a_n$, which must be an integer since the integrality assumption on f passes easily to all its discrete derivatives. But by part (ii), $b_n = n!a_n$, so b_n is an integer. Now induction along with integrality of $\binom{m}{n}$ finishes the proof.)

B6. (i) See the figure below. Two circles G_1, G_2 intersect at points X, Y. Choose two other points A, B on G_1 as shown in the figure. The line segment from A to X is extended to intersect G_2 at point L. The line segment from L to Y is extended to meet G_1 at point C. Likewise the line segment from B to Y is extended to meet G_2 at point M and the segment from M to X is extended to meet G_1 at point D. Show that AB is parallel to CD.



(ii) See the figure below. A triangle CDE is given. A point A is chosen between D and E. A point B is chosen between C and E so that AB is parallel to CD. Let F denote the point of intersection of segments AC and BD. Show that the line joining E and F bisects both segments AB and segment CD. (Hint: You may use Ceva's theorem. Alternatively, you may additionally assume that the trapezium $ABCD$ is a cyclic quadrilateral and proceed.)



(iii) Using parts (i) and (ii) describe a procedure to do the following task: given two circles G_1 and G_2 intersecting at two points X and Y determine the center of each circle *using only a straightedge*. *Note:* Recall that a straightedge is a ruler without any markings. Given two points A, B, a straightedge allows one to construct the line segment joining A, B. Also, given any two non-parallel segments, we can use a straightedge to find the intersection point of the lines containing the two segments by extending them if necessary.

Answer (i): Draw segment BD. Now $\angle BDC = \angle BYC = \angle LYM = \angle LXM = \angle AXD = \angle ABD$, where the second and the fourth equalities are due to opposite angles and the other three equalities due to angles being in the same arc. Therefore AB and CD are parallel.

Answer (ii): Let line EF meet segment CD in point H and segment AB in point I. By Ceva's theorem in $\triangle CDE$, we have $\frac{DA}{AE}\frac{EB}{BC}\frac{CH}{HD} = 1$. As AB and CD are parallel, $\frac{DA}{AE} = \frac{BC}{EB}$, so $CH = DH$. Also by the basic proportionality theorem, $\frac{AI}{DH} = \frac{AE}{DE} = \frac{BE}{CE} = \frac{BI}{CH}$ and since $CH = DH$, $AI = BI$. (If you assume additionally that $ABCD$ is cyclic, it is easy to see using equality of angles in the same arc and of alternate angles made by a transversal that the triangles DEC and DFC are isosceles and in fact line EF is the perpendicular bisector of segments CD and AB.)

Answer (iii): Extend AD and BC to meet in E and take $F =$ the point of intersection of AC and BD. By parts (i) and (ii), the line EF is the bisector of two parallel chords and hence contains a diameter of the circle G_1. Repeat the procedure with some other points A' and B' on G_1 to get another diameter of G_1. The intersection of the two diameters is the center of G_1. Repeat the procedure for G_2.

Note: If *lines* AD and BC do not meet, they are parallel. Then $ABCD$ must be a rectangle (why?) and its diagonals are diameters, which intersect in the centre of G_1. Note that here we have to assume that we can decide if two lines are parallel, which is implicit in the given assumption that if two lines intersect, then we can actually find the point of intersection by extending the given finite segments.