Entrance Examination for CMI BSc (Mathematics & Computer Science) May 2011

Attempt all problems from parts A and C. Attempt any 7 problems from part B.

Part A. Choose the correct option and explain your reasoning briefly. Each problem is worth 3 points.

1. The word MATHEMATICS consists of 11 letters. The number of distinct ways to rearrange these letters is
(A) 11! (B) $\frac{11!}{3}$ (C) $\frac{11!}{6}$ (D) $\frac{11!}{8}$

2. In a rectangle ABCD, the length BC is twice the width AB. Pick a point P on side BC such that the lengths of AP and BC are equal. The measure of angle CPD is
(A) 75° (B) 60° (C) 45° (D) none of the above

3. The number of θ with $0 \leq \theta < 2\pi$ such that $4\sin(3\theta + 2) = 1$ is
(A) 2 (B) 3 (C) 6 (D) none of the above

4. Given positive real numbers $a_1, a_2, \ldots, a_{2011}$ whose product $a_1 a_2 \cdots a_{2011}$ is 1, what can you say about their sum $S = a_1 + a_2 + \cdots + a_{2011}$?
(A) S can be any positive number.
(B) $1 \leq S \leq 2011$.
(C) $2011 \leq S$ and S is unbounded above.
(D) $2011 \leq S$ and S is bounded above.

5. A function f is defined by $f(x) = e^x$ if $x < 1$ and $f(x) = \log_e(x) + ax^2 + bx$ if $x \geq 1$. Here a and b are unknown real numbers. Can f be differentiable at $x = 1$?
(A) f is not differentiable at $x = 1$ for any a and b.
(B) There exist unique numbers a and b for which f is differentiable at $x = 1$.
(C) f is differentiable at $x = 1$ whenever $a + b = e$.
(D) f is differentiable at $x = 1$ regardless of the values of a and b.

6. The equation $x^2 + bx + c = 0$ has *nonzero* real coefficients satisfying $b^2 > 4c$. Moreover, exactly one of b and c is irrational. Consider the solutions p and q of this equation.
(A) Both p and q must be rational.
(B) Both p and q must be irrational.
(C) One of p and q is rational and the other irrational.
(D) We cannot conclude anything about rationality of p and q unless we know b and c.

7. When does the polynomial $1 + x + \cdots + x^n$ have $x - a$ as a factor? Here n is a positive integer greater than 1000 and a is a *real* number.
(A) if and only if $a = -1$
(B) if and only if $a = -1$ and n is odd
(C) if and only if $a = -1$ and n is even
(D) We cannot decide unless n is known.

Part B. Attempt any 7 problems. Explain your reasoning. Each problem is worth 7 points.

1. In a business meeting, each person shakes hands with each other person, with the exception of Mr. L. Since Mr. L arrives after some people have left, he shakes hands only with those present. If the total number of handshakes is exactly 100, how many people left the meeting before Mr. L arrived? (Nobody shakes hands with the same person more than once.)

2. Show that the power of x with the largest coefficient in the polynomial $(1 + \frac{2x}{3})^{20}$ is 8, i.e., if we write the given polynomial as $\sum_i a_i x^i$ then the largest coefficient a_i is a_8.

3. Show that there are infinitely many perfect squares that can be written as a sum of six consecutive natural numbers. Find the smallest such square.

4. Let S be the set of all 5-digit numbers that contain the digits 1,3,5,7 and 9 exactly once (in usual base 10 representation). Show that the sum of all elements of S is divisible by 11111. Find this sum.

5. It is given that the complex number $i - 3$ is a root of the polynomial $3x^4 + 10x^3 + Ax^2 + Bx - 30$, where A and B are unknown real numbers. Find the other roots.

6. Show that there is no solid figure with exactly 11 faces such that each face is a polygon having an odd number of sides.

7. To find the volume of a cave, we fit X, Y and Z axes such that the base of the cave is in the XY-plane and the vertical direction is parallel to the Z-axis. The base is the region in the XY-plane bounded by the parabola $y^2 = 1 - x$ and the Y-axis. Each cross-section of the cave perpendicular to the X-axis is a square.

(a) Show how to write a definite integral that will calculate the volume of this cave.
(b) Evaluate this definite integral. Is it possible to evaluate it without using a formula for indefinite integrals?

8. $f(x) = x^3 + x^2 + cx + d$, where c and d are real numbers. Prove that if $c > \frac{1}{3}$, then f has exactly one real root.

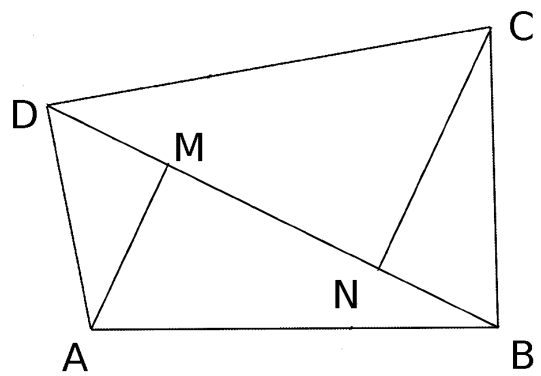
9. A real-valued function f defined on a closed interval $[a, b]$ has the properties that $f(a) = f(b) = 0$ and $f(x) = f'(x) + f''(x)$ for all x in $[a, b]$. Show that $f(x) = 0$ for all x in $[a, b]$.

Part C. Explain your reasoning. Each problem is worth 10 points.

1. Show that there are exactly 16 pairs of integers (x, y) such that $11x + 8y + 17 = xy$. You need not list the solutions.

2. A function g from a set X to itself satisfies $g^m = g^n$ for positive integers m and n with $m > n$. Here g^n stands for $g \circ g \circ \cdots \circ g$ (n times). Show that g is one-to-one if and only if g is onto. (Some of you may have seen the term "one-one function" instead of "one-to-one function". Both mean the same.)

3. In a quadrilateral ABCD, angles at vertices B and D are right angles. AM and CN are respectively altitudes of the triangles ABD and CBD. See the figure below. Show that BN = DM.



In this figure the angles ABC, ADC, AMD and CNB are right angles.