Chennai Mathematical Institute

Entrance Examination for B.Sc. (Mathematics & Computer Science) May 2010

Duration: 3 hours Maximum Score: 100

PART A

Instructions:

- There are 13 questions in this part. Each question carries 4 marks.
- Answer all questions.

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1. Find all $x \in [-\pi, \pi]$ such that $\cos 3x + \cos x = 0$.

2. A polynomial $f(x)$ has integer coefficients such that $f(0)$ and $f(1)$ are both odd numbers. Prove that $f(x) = 0$ has no integer solutions.

3. Evaluate:

   (a) $\displaystyle\lim_{x\to 1} \frac{n - \sum_{k=1}^{n} x^k}{1-x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{e^{-1/x}}{x}$

4. Show that there is no infinite arithmetic progression consisting of distinct integers all of which are squares.

5. Find the remainder given by $3^{89} \times 7^{86}$ when divided by 17.

6. Prove that

$$\frac{2}{0!+1!+2!} + \frac{3}{1!+2!+3!} + \cdots + \frac{n}{(n-2)!+(n-1)!+n!} = 1 - \frac{1}{n!}$$

7. If a, b, c are real numbers > 1, then show that

$$\frac{1}{1+\log_{a^2b}\frac{c}{a}} + \frac{1}{1+\log_{b^2c}\frac{a}{b}} + \frac{1}{1+\log_{c^2a}\frac{b}{c}} = 3$$

8. If 8 points in a plane are chosen to lie on or inside a circle of diameter 2cm then show that the distance between some two points will be less than 1cm.

9. If $f(x) = \dfrac{x^n}{n!} + \dfrac{x^{n-1}}{(n-1)!} + \cdots + x + 1$, then show that $f(x) = 0$ has no repeated roots.

10. Given $\cos x + \cos y + \cos z = \dfrac{3\sqrt{3}}{2}$ and $\sin x + \sin y + \sin z = \dfrac{3}{2}$ then show that $x = \dfrac{\pi}{6} + 2k\pi,\ y = \dfrac{\pi}{6} + 2\ell\pi,\ z = \dfrac{\pi}{6} + 2m\pi$ for some $k, \ell, m \in \mathbf{Z}$.

11. Using the fact that $\sqrt{n}$ is an irrational number whenever n is not a perfect square, show that $\sqrt{3}+\sqrt{7}+\sqrt{21}$ is irrational.

12. In an isoceles $\triangle$ABC with A at the apex the height and the base are both equal to 1cm. Points D, E and F are chosen one from each side such that BDEF is a rhombus. Find the length of the side of this rhombus.

13. If b is a real number satisfying $b^4 + \frac{1}{b^4} = 6$, find the value of $\left(b + \frac{i}{b}\right)^{16}$ where $i = \sqrt{-1}$.

PART B

Instructions:

- There are seven questions in this part. Each question carries 8 marks.
- Answer any six questions.

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1. Let $a_1, a_2, ..., a_{100}$ be 100 positive integers. Show that for some m, n with $1 \leq m \leq n \leq 100$, $\sum_{i=m}^{n} a_i$ is divisible by 100.

2. In $\triangle$ ABC, BE is a median, and O the mid-point of BE. The line joining A and O meets BC at D. Find the ratio $\overline{\text{AO}} : \overline{\text{OD}}$ (Hint: Draw a line through E parallel to AD.)

3. (a) A computer program prints out all integers from 0 to ten thousand in base 6 using the numerals 0,1,2,3,4 and 5. How many numerals it would have printed?
   (b) A 3-digt number *abc* in base 6 is equal to the 3-digit number *cba* in base 9. Find the digits.

4. (a) Show that the area of a right-angled triangle with all side lengths integers is an integer divisible by 6.
   (b) If all the sides and area of a triangle were rational numbers then show that the triangle is got by 'pasting' two right-angled triangles having the same property.

5. Prove that $\int_1^b a^{\log_b x}\, dx > \ln b$ where $a, b > 0, b \neq 1$.

6. Let C_1, C_2 be two circles of equal radii R. If C_1 passes through the centre of C_2 prove that the area of the region common to them is $\frac{R^2}{6}(4\pi - \sqrt{27})$.

7. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arithmetic progressions. Prove that the points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are collinear.