

Instructions

- Enter your *Registration Number* here: **CMIPG ID**

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Enter your *Examination Centre* here:

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- The time allowed is 3 hours.
 - Total Marks: 100
 - Each question carries 20 marks.
- Answer all questions.
 - *Rough Work*: The coloured blank pages are to be used for rough work only.
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For office use only

1	
2	
3	
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Total	

CMI Ph. D. Physics Entrance Exam 2016

(1) A particle is constrained to move on a parabola $y = \alpha x^2$ in the plane where α is a constant of dimension inverse length. Let there be a constant gravitational force in negative z direction.

- (a) Write down the suitable generalized co-ordinates for this system. [1 Mark]
(b) Write down the Lagrangian and obtain Lagrangian equations of Motion. [1 Mark]
(c) Find the equilibrium position of the particle and write the equation for small oscillations about this equilibrium. [1 Mark]
(d) By solving the above equation show that particle executes simple harmonic motion about the equilibrium position. [2 Marks]

(2) Consider three Newtonian particles of masses m_1, m_2, m_3 which interact with each other only through Newtonian gravitational force. Let their position vectors be \vec{r}_1, \vec{r}_2 , and \vec{r}_3 respectively with respect to the center of mass of the system.

- (a) Write down the equations of motion for the system. [4 Marks]
(b) Assuming that the distance d between any pair of masses is equal and is in fact constant, show that the system rotates with a constant angular velocity in its plane and about the center of mass. [6 Marks]

(3) Consider a uniform hoop of mass M and radius R which hangs in a vertical plane supported by a hinge at a point on the hoop as shown in the figure. Calculate the natural frequency of small oscillations. [5 Marks]

(4) Recall that under a gauge transformation in magnetostatics, the vector potential is transformed to $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) - \nabla\chi(\mathbf{r})$ where $\chi(\mathbf{r})$ is a scalar function. [20 Marks]

(a) Show that Ampere's law (in differential form) takes the same form for both \mathbf{A} and \mathbf{A}' . [2 Marks]

(b) Give the differential equation that χ must satisfy so that \mathbf{A}' is in Coulomb gauge. Write an integral expression for its solution, assuming $\mathbf{A} \rightarrow 0$ sufficiently fast as $|\mathbf{r}| \rightarrow \infty$. Proceed by first writing Poisson's equation for the electrostatic potential $\phi(\mathbf{r})$ due to a localized charge distribution $\rho(\mathbf{r})$ and its solution via an integral. Give an appropriate analogy between the two problems. [5 Marks]

(c) Find the magnetic field $\mathbf{B}(\mathbf{r})$ represented by the vector potential $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{b})$ where \mathbf{b} is a constant vector. Hint: For two vector fields we have $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$. [5 Marks]

(d) Find whether the above vector potential $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{b})$ is in Coulomb gauge or not. Hint: $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$. [4 Marks]

(e) Suppose the constant vector $\mathbf{b} = b_0 \hat{z}$ points along the z axis. Find the vector potential \mathbf{A} . Roughly plot the resulting vector field \mathbf{A} in the $x - y$ plane. [4 Marks]

(5)

(a) Consider a particle of mass m in a harmonic oscillator potential of frequency ω . Suppose the particle is in the state

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

where

$$c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$$

and α is a complex number. Show that $|\alpha\rangle$ is an eigenstate of the annihilation operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$. [3]

(b) Show that the probability $P_n(\alpha)$ of finding the state $|\alpha\rangle$ to have the value n when the operator $\hat{a}^\dagger \hat{a}$ is measured is given by the Poisson distribution:

$$P_n(\alpha) = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \quad [2]$$

(c) Show that in the state $|\alpha\rangle$, $\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}} \mathbf{Re}(\alpha)$ and $\langle p \rangle = \sqrt{2m\hbar\omega} \mathbf{Im}(\alpha)$. [2]

(d) Show that the state $|\alpha\rangle$ is a minimum uncertainty state *i.e.*

$$\Delta x \Delta p = \frac{\hbar}{2} \quad [3]$$

(e) If at a time $t = 0$, the state is $|\psi(0)\rangle = |\alpha\rangle$, show that at a later time

$$|\psi(t)\rangle = e^{-\frac{i\omega t}{2}} |\alpha e^{-i\omega t}\rangle$$

What is the origin of the phase factor? [3]

(f) From (c) and (e) show that as functions of time, $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ follow the classical trajectory of the harmonic oscillator. [2]

(g) Starting from $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$, show that the position space wave function for this state is $\psi_\alpha(x) = e^{i\langle \hat{p} \rangle x / \hbar} u_0(x - \langle \hat{x} \rangle)$, where $u_0(x)$ is the ground state gaussian function. [5]

(6) Consider a system of N free spin half particles of mass m obeying Fermi Dirac statistics at temperature $T=0$ and occupying a volume V .

(a) Find the Fermi energy E_F of the system. [3 Marks]

(b) How many particles have energy $E \leq E_F$? [3 marks]

(c) Calculate the average value of U^3 where U is the speed of the particle. [4 marks]

(7). Consider a ideal gas having N distinguishable particles occupying a volume V . Consider a small volume V_1 .

(a) What is the probability that the volume V_1 contains r particles? [3 marks]

(b) In the limit of large N show that r is maximum for $r_0 = N V_1 / V$. [3 marks]

(c) Show that the probability distribution of $r - r_0$ is a gaussian [4 marks]

You can use the Stirling's approximation $\ln N! = N \ln N - N$ for the parts (b),(c) .

(8) Consider the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4 \cos \theta}$$

(a) Define $Z = e^{i\theta}$ and express the integrand as a function of Z . [2 Marks]

(b) Identify the singularities of the above function in the complex Z plane. [2 Marks]

(c) Identify the contour of integration (in the complex Z plane) and perform the integral using the residue theorem. [6 marks]

(9) Given

$$\begin{aligned}\lambda(t) &= 1 - |t|, \quad |t| \leq 1 \\ &= 0, \quad \text{otherwise.}\end{aligned}$$

(a) Plot $\lambda(t)$

[2 Marks]

(b) Find the Fourier transform $\tilde{\lambda}(\omega)$ of $\lambda(t)$.

[6 Marks]

(c) Plot $\tilde{\lambda}(\omega)$.

[2 Marks]