Physics Phd. Entrance 2016 : Solutions

April 19, 2017

1 Classical Mechanics

Problem 1

Errata : Constant gravtitational force is in the negative y direction. There is no z-direction in the problem.

Problem 1(a) : Generalized co-ordinates, $x,\ \dot{x}$.

Problem 1(b) : $L \ = \ \frac{m}{2}(\dot{x}^2 + 2\alpha\ x\dot{x}) + mg\alpha x^2$.

Problem 1(c) : Equillibrium position is : $2mpg\alpha x \ = \ 0 \ \implies \ x = 0, y = 0$.

Problem 1(d) : For small oscillations, one can neglect $x^2\dot{x}^2$ term and whence eq. of motion in this case is

$$\frac{d^2x}{dt^2} \ = \ -2\ g\ \alpha\ x \tag{1}$$

Whence x executes simple Harmonic oscillation with frequency, $\sqrt{2g\alpha}$.

Problem 2

Problem 2(a) : $m_i\vec{a}_i = \sum_{j\neq i} \frac{Gm_jm_i}{|r_{ij}|^2}\hat{r}_{ij}$

Problem 2(b) : As $a_i = \frac{1}{d^2}\frac{1}{m_i}\sum_{j\neq i} Gm_jm_i\hat{r}_{ij} \implies \sum_i \vec{a}_i \times \vec{r}_i = 0$, Hence the proof.

Problem 3

For small oscillation around the axis passing through the Hinge, we have following equation for $\tau = I\frac{d^2\theta}{dt^2}$. $\frac{MR^2}{2}\frac{d^2\theta}{dt^2} = MGR\theta \implies \sqrt{\frac{2G}{R}}$ is frequency of small oscillation.

2 Quantum Mechanics

a)

$$\hat{a} \mid \alpha >= \sum_{n=0}^{\infty} c_n \hat{a} \mid n >= \sum_{n=0}^{\infty} c_n \sqrt{n} \mid n-1 >= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} \mid n >$$

Also

$$c_{n+1}\sqrt{n+1} = e^{-|\alpha|^2/2} \frac{\alpha^{n+1}\sqrt{n+1}}{\sqrt{n+1}!} = \alpha c_n \Rightarrow \hat{a} \mid \alpha >= \alpha \sum_{n=0}^{\infty} c_n \mid n >= \alpha \mid \alpha >$$

b)

$$P_n = \mid < n \mid \alpha > \mid^2 = \mid c_n \mid^2 = \mid e^{-|\alpha|^2/2} \alpha^n / \sqrt{n}! \mid^2 = e^{-|\alpha|^2} (\mid \alpha \mid^2)^n / \sqrt{n}!$$

c)

$$\hat{x} = \sqrt{\frac{2\hbar}{m\omega}} \frac{\hat{a} + \hat{a}^\dagger}{2}, \quad \hat{p} = \sqrt{2m\hbar\omega} \frac{\hat{a} - \hat{a}^\dagger}{2i}$$

Also

$$\hat{a} \mid \alpha >= \alpha \mid \alpha > \Rightarrow < \alpha \mid a \mid \alpha >= \alpha, \qquad < \alpha \mid a^\dagger \mid \alpha >= \alpha^*$$

It follows that

$$< \hat{x} >= \sqrt{\frac{2\hbar}{m\omega}} \mathbf{Re}(\alpha), \qquad < \hat{p} >= \sqrt{2m\hbar\omega} \mathbf{Im}(\alpha)$$

d)

$$\Delta x = \sqrt{< x^2 > - < x >^2} \quad \Delta p = \sqrt{< p^2 > - < p >^2}$$

Substitute for $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ and calculate to get

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}, \quad \Delta p = \sqrt{\frac{m\hbar\omega}{2}} \Rightarrow \Delta x \Delta p = \frac{\hbar}{2}$$

e)

$$| \psi(t) >= U(t) \mid \psi(0) >= \infty_{n=0}^{\infty} c_n U(t) \mid n >= \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} \mid n >$$

Also $E_n = \hbar\omega(n + 1/2)$ and $c_n = e^{-|\alpha|^2/2}\alpha^n/\sqrt{n}!$.
Substituting in the previous equation, we get

$$| \psi(t) >= e^{-i\omega t/2} \mid \alpha(t) >$$

where $\alpha(t) = \alpha e^{-i\omega t}$.
The phase $e^{-i\omega t/2}$ comes from the zero-point energy.
f)

$$<| \psi(t) \mid \hat{x} \mid \psi(t) >=<| \alpha(t) \mid \hat{x} \mid \alpha(t) >= \sqrt{\frac{2\hbar}{m\omega}}\mathbf{Re}(\alpha e^{-i\omega t})$$

Similarly

$$<| \psi(t) \mid \hat{p} \mid \psi(t) >=<| \alpha(t) \mid \hat{p} \mid \alpha(t) >= \sqrt{2m\hbar\omega}\mathbf{Im}(\alpha e^{-i\omega t})$$

g)

$$\alpha < x \mid \alpha >=< x \mid \hat{a} \mid \alpha >=< x \mid A\hat{x} + iB\hat{p} \mid \alpha >= (Ax + Bd/dx) < x \mid \alpha >$$

$$\Rightarrow ()\psi' + ()\psi = \alpha\psi$$

$$\Rightarrow \psi = \psi_0 e^{\sqrt{\frac{2m\omega}{\hbar}}(\alpha x - x^2\sqrt{\frac{m\omega}{8\hbar}})}$$

Electro-Magnetism

1. $\langle \mathbf{20} \rangle$ Recall that under a gauge transformation in magnetostatics, the vector potential is transformed to $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) - \nabla\chi(\mathbf{r})$ where $\chi(\mathbf{r})$ is a scalar function.

   (a) $\langle \mathbf{2} \rangle$ Show that Ampere's law (in differential form) takes the same form for both $\mathbf{A}$ and $\mathbf{A}'$.

   (b) $\langle \mathbf{5} \rangle$ Give the differential equation that χ must satisfy so that $\mathbf{A}'$ is in Coulomb gauge. Write an integral expression for its solution, assuming $\mathbf{A} \to 0$ sufficiently fast as $|\mathbf{r}| \to \infty$. Proceed by first writing Poisson's equation for the electrostatic potential $\phi(\mathbf{r})$ due to a localized charge distribution $\rho(\mathbf{r})$ and its solution via an integral. Give an appropriate analogy between the two problems.

   (c) $\langle \mathbf{5} \rangle$ Find the magnetic field $\mathbf{B}(\mathbf{r})$ represented by the vector potential $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{b})$ where $\mathbf{b}$ is a constant vector. Hint: For two vector fields we have $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$.

   (d) $\langle \mathbf{4} \rangle$ Find whether the above vector potential $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{b})$ is in Coulomb gauge or not. Hint: $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$.

   (e) $\langle \mathbf{4} \rangle$ Suppose the constant vector $\mathbf{b} = b_0 \hat{z}$ points along the z axis. Find the vector potential $\mathbf{A}$. Roughly plot the resulting vector field $\mathbf{A}$ in the $x - y$ plane.

Solutions

1. ANS: with $\mathbf{B} = \nabla \times \mathbf{A}$, Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ becomes $\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$. Now the curl of a gradient vanishes $\nabla \times (\nabla\chi) = 0$, so Ampere's law for $\mathbf{A}'$ is also $\nabla \times (\nabla \times \mathbf{A}') = \mu_0 \mathbf{j}$.

2. ANS: For $\mathbf{A}'$ to be in Coulomb gauge $\nabla \cdot \mathbf{A}' = 0$ we must have $\nabla^2\chi = \nabla \cdot \mathbf{A}$, which is Poisson's equation for χ. We compare it with Poisson's equation for the electrostatic potential $\nabla^2\phi = -\rho/\epsilon_0$ whose solution is $\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'$. So χ is like the electrostatic potential while the source $\nabla \cdot \mathbf{A}$ plays the role of $-\rho/\epsilon_0$. Thus we must have $\chi(\mathbf{r}) = -\frac{1}{4\pi} \int \frac{(\nabla \cdot \mathbf{A})(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'$. 1 mark for each part.

3. ANS: $\mathbf{B} = \mathbf{b}$ is a uniform magnetic field. We evaluate $\mathbf{B} = \nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{b})$. Using the fact that $\mathbf{b}$ is a constant vector, we get $\nabla \times \mathbf{A} = -\frac{1}{2}[-\mathbf{b}(\nabla \cdot \mathbf{r}) + (\mathbf{b} \cdot \nabla)\mathbf{r}]$. Now $\mathbf{r} = (x, y, z)$ so $\nabla \cdot \mathbf{r} = \partial_i x_i = 3$ and $(\mathbf{b} \cdot \nabla)\mathbf{r} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} = \mathbf{b}$. Thus $\mathbf{B} = -\frac{1}{2}(-3\mathbf{b} + \mathbf{b}) = \mathbf{b}$. *Alternate approach:* take $\mathbf{b} = (b_1, b_2, b_3)$ with b_1 constants. Then $-2\mathbf{A} = \mathbf{r} \times \mathbf{b} = (yb_3 - zb_2, zb_1 - xb_3, xb_2 - yb_1)$. Then calculate $\nabla \times (\mathbf{r} \times \mathbf{b}) = (-2b_1, -2b_2, -2b_3)$. Thus we see that $\mathbf{B} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{b}) = \mathbf{b}$.

4. ANS: Yes, it is in Coulomb gauge. We calculate $\nabla \cdot \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{b}) = -\frac{1}{2}(\mathbf{b} \cdot \nabla \times \mathbf{r} - \mathbf{r} \cdot \nabla \times \mathbf{b}) = -\frac{1}{2}\mathbf{b} \cdot \nabla \times (\frac{1}{2}\nabla r^2) = 0$. This is because $\mathbf{b}$ is a constant vector and the curl of the gradient $\mathbf{r} = \frac{1}{2}\nabla r^2$ is zero. Thus we have a Coulomb gauge vector potential for a constant/uniform magnetic field.

5. ANS: $\mathbf{A} = -\frac{1}{2}b_0(\mathbf{r} \times \hat{z}) = -\frac{1}{2}b_0(x\hat{x} + y\hat{y} + z\hat{z}) \times \hat{z} = -\frac{1}{2}b_0(-x\hat{y} + y\hat{x}) = \frac{1}{2}b_0(x\hat{y} - y\hat{x}) = \frac{1}{2}rb_0(\cos\theta\hat{y} - \sin\theta\hat{x}) = \frac{1}{2}rb_0\hat{\theta}$. So $\mathbf{A}$ circulates around the z axis or the origin of the $x - y$ plane. It points in the tangential/azimuthal direction counter-clockwise. The field lines (integral curves) of $\mathbf{A}$ are counter-clockwise directed concentric circles. The magnitude of $\mathbf{A}$ grows linearly with radial distance. Here we used $x/r = \cos\theta$ and $y/r = \sin\theta$ in plane polar coordinates.

3 Mathematical Physics

1. Let $z = e^{i\theta}$ (unit circle) (and hence $d\theta = -i\frac{dz}{z}$) so that $\cos\theta = \frac{1+z^2}{2z}$. Hence the integral to be evaluated now reads

$$I = -i\oint \frac{dz}{z\left(5 + 4\left(\frac{z^2+1}{2z}\right)\right)} = -i\oint \frac{dz}{(2z+1)(z+2)},$$

where the integration is now over the unit circle. As is evident, there are two singularities at $z = -1/2$ and $z = 2$. The second singularity is outside the unit circle and the first one ($z = -1/2$) is a simple pole inside the unit circle. Defining

$$\phi(z) = (z + 1/2)\frac{1}{(2z+1)(z+2)} = \frac{1}{2(z+2)},$$

the residue corresponding to this pole is given by

$$\lim_{z\to -1/2} \phi(z) = 1/3$$

Hence using residue theorem one finds

$$I = (-i)\,(2\pi i)(1/3) = \frac{2\pi}{3}$$

2. (a) The Fourier transform of $\lambda(t)$ is defined as

$$\begin{aligned}
\tilde{\lambda}(\omega) &= \int_{-\infty}^{\infty} \lambda(t) e^{-i\omega t}\, dt \\
&= \int_{-1}^{1} \lambda(t) e^{-i\omega t}\, dt \\
&= \int_{-1}^{0} (1+t) e^{-i\omega t}\, dt + \int_{0}^{1} (1-t) e^{-i\omega t}\, dt \\
&= \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^2
\end{aligned}$$

4 Statistical Mechanics

1. Enaergy levels of a single particle in SHO is

$$E_n = \hbar\omega(n + 1/2) \qquad n = 0, 1, 2, \ldots$$

For our case $n = 0, 1, 2, 3$. Thus the possible arrangements are with n_1 and n_2 as the quantum numbers for the two particles. Further $en = E_n/(\hbar\omega)$

sl.n	n_1	n_2	en
1.	0	0	1
2.	0	1	2
3.	0	2	3
4.	0	3	4
5.	1	0	2
6.	1	1	3
7.	1	2	4
8.	1	3	5
9.	2	0	3
10.	2	1	4
11.	2	2	5
12.	2	3	6
13.	3	0	4
14.	3	1	5
15.	3	2	6
16.	3	3	7

Out of these

Sl.nos. 2 and 5 represent the same state

Sl.nos 3 and 9 represent the same state

Sl.nos.4 and 13 represent the same state

Sl.nos. 7 and 10 represent the same state

Sl.nos. 8 and 14 represent the same state

Sl.nos. 12 and 15 represent the same state

Thus the total number of states are 16-6=10.

(b) Energy $= 3\hbar\omega$ occurs twice and the state is doubly degenerate.

(c) Wave functions are

$$u_0(x_1)u_2(x_2) + u_2(x_1)u_0(x_2)$$

and

$$u_1(x_1)u_1(x_2)$$

2. The energy of a magnetic moment in a magnetic field is $-\vec{\mu}.\vec{B} = -gS_zB_0$ if the magnetic field is along z- axis with strength B_0.

Thus the energies of the three states are (for values of $S_z = \hbar\,,\,0\,,\,-\hbar$) are $g\hbar\,,$ 0 and $-g\hbar$.

(a)The average value of magnetic moment is

$$< \mu_z >= g\hbar \left(\frac{(e^{\frac{gB_0\hbar}{kT}} + 0 - e^{\frac{-gB_0\hbar}{kT}})}{(e^{\frac{gB_0\hbar}{kT}} + 1 + e^{\frac{-gB_0\hbar}{kT}})} \right)$$

For small fields this becomes

$$< \mu_z >= \frac{2g^2B_0}{3}$$

the susceptitibilty is

$$\chi = \frac{2}{3}g^2$$

(b) The probability in the state with S_z =1 is

$$P = \frac{e^{\frac{g\hbar B_0}{kI}}}{(e^{\frac{gB_0\hbar}{kT}} + 1 + e^{\frac{-gB_0\hbar}{kT}})}$$