Instructions

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  Enter your *Examination Centre* here: ☐☐☐☐☐☐☐☐☐☐☐☐☐

  - The time allowed is 3 hours.
  - Total Marks: 100
  - Each question carries 20 marks.

- Answer all questions.

- *Rough Work:* The coloured blank pages are to be used for rough work only.

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1	
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Total	

CMI Ph. D. Physics Entrance Exam 2015

(1) A particle of mass m moves under the influence of an attractive central force $f(r)\hat{r}$.

(a) Show that the motion is planar. [3 mks]

(b) The motion of the particle can be treated as one dimensional with the introduction of an appropriate effective potential which involves the angular momentum. Write down the effective potential and total energy of the particle. [4 mks]

(c) Using the result obtained above, show that by proper choice of the initial conditions, you can always get a circular orbit. [5 mks]

(d) If you subject the above circular orbit to a small radial perturbation, obtain the relation between $f(r), r$ and $\frac{\partial f}{\partial r}$ for the orbit to be stable. [4 mks]

(e) If the force is of the form $f(r) = -K/r^n$, obtain the condition on n for the circular orbit to be stable. [4 mks]

(2) Problem (c) is compulsory. Attempt only one of the two problems (a) and (b). Maximum available points are 20.

(a) A ring of radius r has charge q uniformly distributed over it. The ring is placed in the $x-y$ plane with its center at the origin.

(i) What is the electric field at any point along the z-axis. [5 mks]

(ii) Use the above result to show the following: Consider a charge $-q$ which is constrained to move on the z-axis that is perpendicular to the plane of the ring, passing through it's center. Show that the charge will execute simple harmonic motion when placed along the z-axis at a distance $z << r$ from the center of the ring. [5 mks]

(b) Consider two plane infinite sheets with uniform charge densities $\pm\sigma$ which intersect at right angles. Find the magnitude and direction of E everywhere and sketch the electric field lines. [10 mks]

(c) A very long solenoid of radius r and having N turns per unit length carries a current $I = I_0 \sin(\omega t)$ in the counter-clockwise direction. Assume that the axis of the solenoid coincides with the z-axis.

(i) Write the expression for $\vec{B}(t)$ inside the solenoid. [3 mks]

(ii) Assuming that $\vec{B}$ is zero outside the solenoid, write down the expression for electric field $\vec{E}(t)$ inside and outside the solenoid. [7 mks]

(3) Problems (a) and (b) are not related: both are compulsory.

(a) Consider a free particle of mass m in a 1-dimensional box. The potential has the form $V(x) = \infty$, $x < 0, x > b$, $V(x) = 0$, $0 < x < a$, $V(x) = V_0$, $a < x < b$. Find the quantization condition for stationary state energy levels for energies $E > V_0$, and the corresponding wavefunction. In the limit of E much larger than V_0, show that the resulting quantization condition you find reduces to the familiar one for a particle in a box with no internal "step" (i.e. with $V_0 \sim 0$). [10 mks]

(b) Consider a system of two non-interacting harmonic oscillators with Hamiltonian

$$H_0 = \omega_1(a_1^\dagger a_1 + \frac{1}{2}) + \omega_2(a_2^\dagger a_2 + \frac{1}{2}),$$

where $a_i, a_i^\dagger$ are the usual creation-annihilation operators for each of the oscillators.
(i) What is the condition on ω_1, ω_2, for the system to have no degeneracy in any level? [4 mks]
(ii) Now say $\omega_1 = \omega_2$. List the three lowest energy levels and their degeneracies. [6 mks]

(4) The rotational energy levels of a diatomic molecule made of two identical atoms with spin half nuclei is given by

$$E[J] \;=\; \lambda J (J + 1) , \qquad\qquad J = 0, 1, 2, \ldots$$

In answering the following questions you need to consider only the rotational energy levels and the fact that the two nuclei are identical. In particular you need not consider electronic or vibrational levels.

(a) What are the allowed spin states? [2 mks]
(b) How does the spin wave function change under the exchange of the two spins? [2 mks]
(c) How would the rotational wave function change under the exchange of the two spatial co-ordinates of the nuclei? [2 mks]
(d) Which spin states are allowed for $J = 0$ and $J = 1$? [4 mks]
(e) How many states (including spin wave function) are there with energy $E = 0$? [2 mks]
(f) How many states are there with energy (including spin states) $E = 2\lambda$? [2 mks]
(h) What will be relative population of the two states $(J = 0, \ J = 1)$, if the system is in equilibrium at temperature T? [6 mks]

(5) Suppose $\hat{n}$ is a unit vector in three dimensions ($\mathbf{R}^3$) with Cartesian components n_i. Consider the 3×3 matrix A with entries

$$A_{ij} = \int n_i n_j dn,$$

where the integration is over all directions in which the unit vector $\hat{n}$ can point. You may need the relation between Cartesian and spherical polar coordinates $z = r\cos\theta$, $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$.

(a) A is a real symmetric matrix. What does this mean for its eigenvalues? [2 mks]

(b) Write the integration element dn **and** the components n_i in spherical polar coordinates θ, ϕ. [Hint: dn is just the area element on a unit sphere.] [5 mks]

(c) Find an integral expression for the 'expectation value' of A in an arbitrary unit norm 'state' $\mathbf{v}$, $\langle \mathbf{v}|A|\mathbf{v}\rangle = \mathbf{v}^t A \mathbf{v}$. Here $\mathbf{v}$ is a unit column vector in three dimensions $\mathbf{v}^t\mathbf{v} = 1$. In particular, show that $\langle \hat{z}|A|\hat{z}\rangle = 4\pi/3$, where $\hat{z} = (0,0,1)$. [5 mks]

(d) Suppose $\mathbf{v}$ is a unit vector in three dimensions. Argue without detailed calculations that $\langle \mathbf{v}|A|\mathbf{v}\rangle$ is independent of the direction of v. [Hint: How can you 'undo' the effect of a rotation $\mathbf{v} \mapsto R\mathbf{v}$? [4 mks]

(e) The result of part (d) may be interpreted as "the expectation value of A is the same in all states". Find all the eigenvalues and eigenvectors of A. [4 mks]