CMI Ph.D. Phy solutions 2015

(1) (a) Show that angular momentum is conserved by calculating Torque.
(b) We have

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + V(r), \qquad E = \frac{m\dot{r}^2}{2} + V_{\text{eff}},$$

where L is the conserved angular momentum and $f = \frac{-dV}{dr}$. Motion can be considered as one dimensional (only radial motion).
(c) Circular motion in potential V is equal to particle at rest in the equilibrium point of the effective potential. Equilibrium position r_0 is defined by the vanishing of the derivative of V_{eff} w.r.t r. $\frac{\partial V_{\text{eff}}}{\partial r}|_{r_0} = 0$ which implies $\frac{\partial V}{\partial r} = \frac{L^2}{m r^3}$. Together with this, if the initial condition is such that $E = V_{\text{eff}}(r_0)$, then you can get a circular orbit.
(d) For stable circular orbit, you need r_0 to be stable equilibrium point which requires r_0 is the minimum of V_{eff}, hence second derivative of V_{eff} w.r.t r evaluated at r_0 should be greater than 0 which implies $\frac{3L^2}{m r^4} - \frac{\partial f}{\partial r} > 0$ at r_0.
(e) Using the form of the force in the above expression, you get $n < 3$ for the circular orbit to be stable.

(2) (a): (i) We want to determine $\vec{E}$ at a distance z along the axis of the ring. From the symmetry of the problem it is obvious that the field is only along the z-axis. This field is determined by summing the contributions from all the infinitesimal charge segments along the ring.

$$dE_z(0,0,z) = \frac{\frac{q}{2\pi r} d\theta}{4\pi\epsilon_0} \frac{z}{(z^2 + r^2)^{\frac{3}{2}}}$$

This gives,

$$\vec{E}(0,0,z) = \frac{qz}{4\pi\epsilon_0 (r^2 + z^2)^{\frac{3}{2}}} \hat{k}$$

where $\hat{k}$ is the unit vector along the z-axis.

(ii) The force on charge $-q$ along the z-axis is,

$$\vec{F} = -\frac{q^2 z}{4\pi\epsilon_0 (r^2 + z^2)^{\frac{3}{2}}} \hat{k}$$

which at a distance $z \ll r$ is $\vec{F} \approx -\frac{q^2 z}{4\pi\epsilon_0 r^3} \hat{k}$. Hence for small displacements from the center of the ring particle will execute simple harmonic motion.

(b) Recall : Magnitude of the electric field due to a infinite sheet of uniform charge density is $E = \frac{\sigma}{2\epsilon_0}$ with the direction perpendicular to the sheet and the field lines are directed towards or away from the sheet depending on the sign of the charge.
Hence for two perpendicular sheets with charge densities $\pm\sigma$ the magnitude of the electric field is constant everywhere in space and is given by $|\vec{E}| = \frac{\sqrt{2}\sigma}{2\epsilon_0}$ with the field lines shown in the figure.

(c) (i) Inside the solenoid, $\vec{B}$ is uniform everywhere and is directed along the axis of the solenoid. It's magnitude is

$$B(t) = \mu_0 N I_0 \sin(\omega t)$$

(ii) For this part we just have to use $\int_C \vec{E} \cdot dl = -\int_A \frac{\partial \vec{B}}{\partial t} \cdot dS$ where C is a counter-clock wise circular loop around the axis of the solenoid which encompasses an area A.
Hence inside the solenoid at $r < R$, (using above equation and axial symmetry)

$$\vec{E} \cdot 2\pi r = -\pi r^2 \mu_0 N I_0 \omega \cos(\omega t) \hat{e}_\phi$$

Similarly, assuming $B = 0$ outside the solenoid, the electric field is

$$\vec{E} = -\frac{R^2}{2r} \mu_0 N I_0 \omega \cos(\omega t) \hat{e}_\phi$$

(3): (a) The Schrodinger equation for the two regions $I \equiv (0, a)$ and $II \equiv (a, b)$ are

$$\psi_I'' + \frac{2mE}{\hbar^2}\psi_I = 0, \quad \psi_{II}'' + \frac{2m(E - V_0)}{\hbar^2}\psi_{II} = 0; \quad k_1 = \sqrt{\frac{2mE}{\hbar^2}}, \quad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}.$$

The boundary conditions are $\psi_I(0) = 0$, $\psi_{II}(b) = 0$, $\psi_I(a) = \psi_{II}(a)$, $\psi_I'(a) = \psi_{II}'(a)$. This gives the wavefunction

$$\psi_I(x) = A_1 \sin(k_1 x), \qquad \psi_{II}(x) = A_2(\sin(k_2 x) - \tan(k_2 b)\cos(k_2 x)),$$

subject to boundary conditions at $x = a$,

$$A_1 \sin(k_1 a) = A_2(\sin(k_2 a) - \tan(k_2 b)\cos(k_2 a)),$$

$$\text{[quantization condition]} \qquad k_1 \frac{\cos(k_1 a)}{\sin(k_1 a)} = k_2 \frac{\cos(k_2 a) + \tan(k_2 b)\sin(k_2 a)}{\sin(k_2 a) - \tan(k_2 b)\cos(k_2 a)}.$$

In the limit of $E \gg V_0$, we have $k_2 \sim k_1$. Then the quantization condition after simplifying becomes $(\sin^2(k_1 a) + \cos^2(k_1 a))\tan(k_1 b) \sim 0$ which is the usual condition $\sin(k_1 b) = 0$ for a free particle in a box with no internal step.
(b) (i) Levels (n_1, n_2) and (m_1, m_2) with arbitrary integers n_i, m_i are degenerate with the same energy if $n_1\omega_1 + n_2\omega_2 = m_1\omega_1 + m_2\omega_2$, i.e. if $\omega_1 = \frac{m_2 - n_2}{n_1 - m_1}\omega_2 \equiv \frac{p}{q}\omega_2$. Thus if ω_1, ω_2 are not

rational multiples of each other, there is no degeneracy.
(ii) If $\omega_1 = \omega_2$, the 4 lowest energy states are $(0,0)$ $[E = \omega_1]$, $\{(1,0),(0,1)\}$ $[E = 2\omega_1]$, $\{(2,0),(1,1),(0,2)\}$ $[E = 3\omega_1]$, $\{(3,0),(2,1),(1,2),(0,3)\}$ $[E = 4\omega_1]$.

(4) (a) S=0,S=1

(b) S=0 changes sign and S=1 remains the same.

(c) J=0 the spatial wave function does not change sign. J=1 the wave function changes sign.

(d) For J=0, S=0 is allowed and for J=1, S=1 is allowed

(e) One state

(f) Nine states.

(h) The Boltzmann factor is $e^{-2\lambda/(kT)}$ and there are three S=1 states and three J=1 states. Thus

$$\frac{N(J=1)}{N(J=0)} = 9\left(e^{-\frac{2\lambda}{kT}}\right)$$

(5)

(a) Since A is real symmetric, eigenvalues are guaranteed to be real. A can be diagonalised by an orthogonal transformation, it must have real eigenvectors that span the vector space.

(b) Integration element is area element on sphere $dn = \sin\theta d\theta d\phi$. Unit vector is $\hat{n} = \frac{1}{r}(x,y,z)$. Writing it in spherical polar coordinates, $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

(c) $\langle v|A|v\rangle = \int (v\cdot n)^2 dn$. $n\cdot\hat{z} = \cos\theta$ and $\langle\hat{z}|A|\hat{z}\rangle = \int \cos^2\theta\sin\theta d\theta d\phi = 4\pi/3$.

(d) Suppose we rotate $v \to Rv$, then we can undo this rotation by making a change of integration variable $\hat{n} \to R^{-1}\hat{n}$. The area element is rotation invariant. So the integral does not change.

(e) Since A has the same expectation value in every state it must be a multiple of the identity. So $4\pi/3$ is its only eigenvalue with multiplicity three. Since A is a multiple of the identity, all non-zero vectors are eigenvectors of A.