CMI Ph.D. Phy solutions 2014

(1) (a) $V(x) = g(x^2 - a^2)^2$, a has the same dimension as x i.e., length and g has dimensions of energy $\times L^{-4}$ or $MT^{-2}L^{-2}$.
(b) It is a double well potential, symmetric about $x = 0$ with a local maximum at $x = 0$ (where $V = ga^4$) and global minima at $x = \pm a$, where $V = 0$. As $|x| \to \infty$, the potential $V(x) \to \infty$.
(c) $L = T - V = \frac{1}{2}m\dot{x}^2 - g(x^2 - a^2)^2$. Equation of motion is the Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$ or $m\ddot{x} = -4gx(x^2 - a^2)$. x is the dependent variable and t is the independent variable. It is a non-linear ordinary differential equation in x since x appears to the third power.
(d) Time independent means $x(t)$ is independent of time, so $\dot{x} \equiv 0$ and $\ddot{x} \equiv 0$. So $x(x^2 - a^2) = 0$ this happens if $x \equiv 0, \pm a$, and these are the three static solutions. $x = 0$ is unstable since it is at a maximum of the potential, $V''(0) < 0$. $x = \pm a$ are stable since they occur at minima of the potential, $V''(\pm a) > 0$.
(e) Let us consider small oscillations about the stable static solution $x = a$. To study this we expand the potential in a quadratic Taylor polynomial $V(x) = V(a) + V'(a)(x - a) + \frac{1}{2}V''(a)(x - a)^2$ around $x = a$. Note that $V(a) = 0$, $V'(a) = 2g(x^2 - a^2)2x|_{x=a} = 0$ while $V''(x) = 4g(x^2 - a^2) + 4gx \cdot 2x$ so $V''(a) = 8ga^2$. So $H(x,p) = \frac{p^2}{2m} + \frac{1}{2}8ga^2(x - a)^2 + \cdots$. Comparing with $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$ we find that $\omega^2 = 8ga^2/m$ or $\omega = 2a\sqrt{2g/m}$. So the time period is $T = 2\pi/\omega = \frac{\pi}{a}\sqrt{\frac{m}{2g}}$.

(2) (a) Using $\vec{\nabla}.\vec{B} = 0$ we have B_z just inside the slab the same as B_z just outside the slab.
Using $\vec{\nabla} \times \vec{H} = 0$, we have

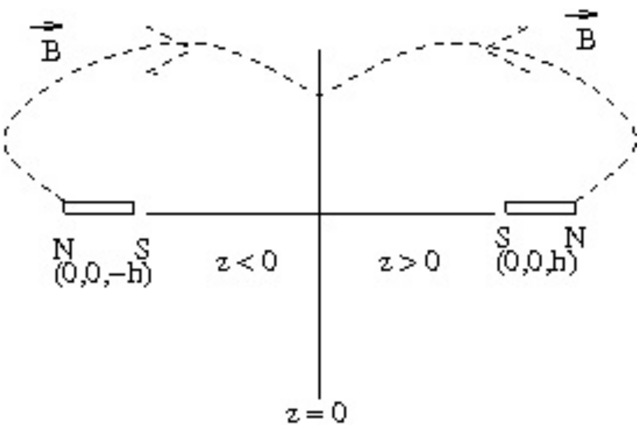
$$\frac{B_x}{\mu_0}|_{z>0, just\ outside} = \frac{B_x}{\mu}|_{z<0, just\ inside}$$

and

$$\frac{B_y}{\mu_0}|_{z>0, just\ outside} = \frac{B_y}{\mu}|_{z<0, just\ inside}$$

(b) Since $B_z = 0$ inside the slab, $B_z = 0$ just outside the slab.

(c) The z- component of the magnetic field produced by the mirror image kept at (0,0,-h) should cancel the z- component of the field produced by the magnet kept at (0,0,h) on the surface $z = 0$. This implies the magnetic moment of the mirror image should be $-m\,\hat{k}$. See figure.



(d) Since there is a magnetic field in the x- an y directions on the $z > 0$ side of the slab, and no field inside the slab, there is a surface current in order to satisfy

$$\vec{\nabla} \times \vec{B} = \vec{J} \text{ (equation)A}$$

where $\vec{J}$ is the total current density.

Using the above equation (A) we have a surface current along the -y direction.

(3) (a) This is a 2-particle system and both spins are non-interacting if $J = 0$. Each of the two spins can be in either of the two σ^z-eigenstates $|\uparrow\rangle, |\downarrow\rangle$ with eigenvalues ± 1 respectively. There are four states $\{ |\uparrow_1, \uparrow_2\rangle, |\uparrow_1, \downarrow_2\rangle, |\downarrow_1, \uparrow_2\rangle, |\downarrow_1, \downarrow_2\rangle \}$. The operators $\sigma^z_{1,2}$ act on the two different Hilbert spaces $\mathcal{H}_{1,2}$ respectively. It can then be checked that the $|\downarrow_1, \downarrow_2\rangle$ has the lowest energy expectation value,

$$\langle \downarrow_1, \downarrow_2 | \mu B(\sigma^z_1 + \sigma^z_2) | \downarrow_1, \downarrow_2\rangle = \mu B[\langle \downarrow_1 | \sigma^z_1 | \downarrow_1\rangle + \langle \downarrow_2 | \sigma^z_2 | \downarrow_2\rangle] = -2\mu B,$$

the other 3 states with similar calculation giving higher $\langle E\rangle$. Roughly both independent spins are aligned by the magnetic field.

(b) With $B = 0$, we need to find the ground states of $-J\sigma^x_1\sigma^x_2$. Representing $\sigma^z = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ with $|\uparrow\rangle = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$, $|\downarrow\rangle = \left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)$, we have $\sigma^x = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ with eigenstates $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. The 2-particle states are $\{ |+_1, +_2\rangle, |+_1, -_2\rangle, |-_1, +_2\rangle, |-_1, -_2\rangle \}$. It is then clear that both states $|+_1, +_2\rangle$, $|-_1, -_2\rangle$ have lower energy: e.g.

$$\langle +_1, +_2|(-J\sigma^x_1\sigma^x_2)|+_1, +_2\rangle = -J\langle +_1|\sigma^x_1|+_1\rangle\langle +_2|\sigma^x_2|+_2\rangle = -J.$$

(c) In first order perturbation theory, the energy change is $\langle\psi_0|\Delta V|\psi_0\rangle$. Taking $|\psi_0\rangle = |\downarrow_1, \downarrow_2\rangle$ and $\Delta V = -J\sigma^x_1\sigma^x_2$, we see that the energy change is $\langle\psi_0|\Delta V|\psi_0\rangle = 0$, since $\langle \downarrow | \sigma^x | \downarrow\rangle = 0$.

(4) (a) Let $T = 0K$. All, N, free nonrelativistic spin-1/2 particles, of mass m, will be uniformly distributed, 2 per state (because of degeneracy due to spin), up to the Fermi energy $\epsilon_F = p_F^2/2m$.

There will be two particles per an area $2\pi\hbar$ of the phase space in each dimension (recall the uncertainty principle). Thus, we have

$$N = \frac{2}{4\pi^2\hbar^2} \int\int\int\int dx dy dp_x dp_y, \quad \text{with } \ p^2/2m \leq \epsilon_F.$$

With $\int\int dxdy = A$, let $N/A = n$. Then, going over to polar coordinates in momentum space, we get

$$n = \frac{1}{\pi\hbar^2} \int_0^{\sqrt{2m\epsilon_F}} pdp = \frac{m\epsilon_F}{\pi\hbar^2}.$$

Thus, we have for the Fermi energy

$$\epsilon_F = \frac{n\pi\hbar^2}{m}.$$

(b) When $T \neq 0$, particles will be distributed according to the Fermi-Dirac distribution. This modifies the equation for n above to

$$n = \frac{1}{\pi\hbar^2} \int_0^\infty \frac{pdp}{e^{\beta(\epsilon-\mu)}+1},$$

where $\epsilon = p^2/2m$, energy of the particle of momentum $\vec{p}$, $\beta = 1/kT$ with k as the Boltzmann constant, and μ is the chemical potential. Now, letting $x = \beta\epsilon$, the above integral for n can be evaluated easily to give

$$n = \frac{m}{\beta\pi\hbar^2} \int_0^\infty \frac{dx}{e^{x-\beta\mu}+1} = \frac{m}{\beta\pi\hbar^2} \ln(1 + e^{\beta\mu}).$$

Equating the two expressions for n and solving for μ we get the required result:

$$\mu = \frac{1}{\beta} \ln(e^{\beta\epsilon_F} - 1).$$

(c) It is easy to see that, as $T \to 0$,

$$\lim_{\beta\to\infty} \mu = \epsilon_F.$$

(5) (a) Assume series solution of the form

$$y(x) = \sum_{n=0}^\infty a_n \, x^n$$

Substituting this solution in the differential equation we obtain the recurrence relation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+3)}$$

The initial condition $y(0) = 0$ implies $a_0 = 1$ and $y'(0) = 1$ implies $a_1 = 0$. Hence the solution is

$$y(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \frac{\sin x}{x}$$

(b) Re-write the expression of the surface in the form $\mathbf{x}^T A\mathbf{x} = 1$, where A is a symmetric matrix and $\mathbf{x}^T$ corresponds to the transopose of $\mathbf{x}$. In this case $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \text{x} \\ \text{y} \\ \text{z} \end{bmatrix}$

The quadratic surface $\mathbf{x}^T A\,\mathbf{x} = 1$ has the principal axes along the three mutually perpendicular Eigenvectors of A and the squares of the principal radii given by λ_i^{-1} (where $i = 1, 2, 3$) represent the three Eigenvalues of A.

Now it amounts to calculating the Eigenvalues and Eigenvectors of A. It is straightforward to obtain the Eigenvalues to be $\lambda = 3, 1, 0$. Based on the above argument, this implies that along the third principal axis, the surface is infinite in extent as the Eigenvalue is zero. Along the plane orthogonal to the third principal axis, it is an ellipse whose semi-major and semi-minor axis are $\frac{1}{\sqrt{3}}$ and 1 respectively. Hence the surface is an elliptic cylinder. The orientation of the principal axes are given by the Eigenvectors which for the Eigenvalues $\lambda = 3, 1, 0$ are

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \text{-1} \\ 0 \\ 1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 1 \\ \text{-1} \\ 1 \end{bmatrix}.$$

(v_3 is the axis of the cylinder infinite in extent.)