1. \( V(\theta) = -mg \cos \theta \) for \(-\pi \leq \theta \leq \pi\). When \( \theta = 0 \), the potential is a minimum at \(-mg\). Plot is that of negative cosine function. Lagrangian \( L = \frac{1}{2} ml^2 \dot{\theta}^2 + mg \cos \theta \).

2. \( ml^2 \ddot{\theta} + mg \sin \theta = 0 \) or \( \ddot{\theta} = -(g/l) \sin \theta \). Static solution means \( \theta \) is constant in time or \( \dot{\theta} \equiv 0 \). This happens iff \( \sin \theta = 0 \) which means \( \theta = 0, \pm \pi \), in other words the bob is at rest at the bottom or at the top. Former is stable equilibrium and latter unstable.

3. Phase space variables are \( \theta, p_{\theta} \). \( p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \). Solve for \( \dot{\theta} \) in terms of \( p_{\theta} \) and substitute in

\[
H(\theta, p_{\theta}) = p_{\theta} \dot{\theta} - L(\theta, \dot{\theta})
\]

to find the hamiltonian

\[
H(\theta, p_{\theta}) = \frac{p_{\theta}^2}{2ml^2} - mg \cos \theta.
\]

4. The phase space is an infinite cylinder \(-\pi \leq \theta \leq \pi\) with \( \theta = \pm \pi \) being identified and \( p_{\theta} \) can be an arbitrary real number.

5. Maximum angle of deflection for any fixed energy in the given range is \( \theta_o = \arccos \left( \frac{-E}{mg} \right) \). Integrate energy equation for \( \dot{\theta} \), \( E = \frac{1}{2} ml^2 \dot{\theta}^2 - mg \cos \theta \) once to get \( T = 4 \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2E + mg \cos \theta}} \). Factor of 4 as energy is even function of \( \dot{\theta} \) and \( \theta \), bob goes back and forth and must return to original position with the original velocity, so the trajectory may be broken into 4 parts each taking the same time.

6. As \( E \rightarrow mg \), \( \theta_o \rightarrow \pm \pi \) and \( T \rightarrow \infty \) since the integrand \( \sim 1/\cos(\theta/2) \) has a simple pole at \( \theta = \pm \pi \). If bob is raised to initial angles approaching arbitrarily close to \( \pm \pi \), then the period of oscillation can be made arbitrarily long.

[I]

1. The possible outcomes are \( S_z = 2 \) and \( S_z = 1 \). The respective probabilities are

\[
\frac{a^2}{a^2 + b^2} \text{ and } \frac{b^2}{a^2 + b^2}
\]

2. Dimensions of \( \lambda \) and \( \mu \) are the same. The dimension is \( (ML^2)^{-1} \) as we know the dimensions of \( S^2 \) and \( S_z^2 \) are those of the square of angular momentum \( (ML^2/T^2) \) and that of \( H \) is energy \( ML^2/T^2 \).

3. The conserved quantities are \( H \), \( S^2 \) and \( S_z \).

4. The expectation value of the first term is \( 6\hbar^2 \) as \( |\psi> \) is an eigenfunction of \( S^2 \) (which are given by \( S(S + 1) \hbar^2 \)) with \( S = 2 \)

The expectation value for \( S_z^2 \), needed for the second term in the Hamiltonian can be found by writing \( S_x = (S_+ + S_-)/2 \) and using the relations

\[
S_+ |S, m> = \hbar \sqrt{(S - m)(S + m + 1)} |S, m + 1 >
\]

and

\[
S_- |S, m> = \hbar \sqrt{S + m(S - m + 1)} |S, m - 1 >
\]
The result is
\[ <\psi | \mu S^2 \| \psi > = \mu \hbar^2 (1 + \frac{3|b|^2}{2(|a|^2 + |b|^2)}) \]

Finally we get
\[ <\psi | H \| \psi > = 6 \lambda \hbar^2 + \mu \hbar^2 (1 + \frac{3|b|^2}{2(|a|^2 + |b|^2)}) \]

[III]

1. Method: Find Eigen vectors of A, Diagonalize A \((D = S^{-1} A S)\) where \(S\) is the matrix formed from Eigen vectors) and take the square root of the diagonal elements of \(D\) \((\sqrt{D})\) and transform it back \(\sqrt{A} = S \sqrt{D} S^{-1}\).

Eigen values: \(\lambda = 3, 1\)

\[ S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \]

Finally

\[ \sqrt{A} = \begin{pmatrix} 1 & -1 + \sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix} \]

2. Soln:

The Fourier Transform of the given function is given by

\[ \tilde{f}(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i k \cdot r}}{r^2 + \lambda^2} \, dx \, dy \, dz. \]  \hspace{1cm} (1)

Introducing spherical polar coordinates

\[ x(r, \theta, \phi) = r \cos \phi \sin \theta \]
\[ y(r, \theta, \phi) = r \sin \phi \sin \theta \]
\[ z(r, \theta, \phi) = r \cos \theta \] \hspace{1cm} (2)

\[ \tilde{f}(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i k \cdot r}}{r^2 + \lambda^2} \, dx \, dy \, dz. \] \hspace{1cm} (3)
Introducing spherical polar coordinates
\[ x(r, \theta, \phi) = r \cos \phi \sin \theta \]  \hspace{1cm} (4)
\[ y(r, \theta, \phi) = r \sin \phi \sin \theta \]  \hspace{1cm} (5)
\[ z(r, \theta, \phi) = r \cos \theta \]  \hspace{1cm} (6)
immediately leads to
\[ dx dy dz = r^2 dr \sin \theta d\theta d\phi \]  \hspace{1cm} (7)
where \( 0 \leq \phi < 2\pi, 0 \leq \theta < \pi, r \geq 0 \).
We also have \( k \cdot r = kr \cos \theta \) where \( k = ||k|| \).
The integral now takes the form
\[ \tilde{f}(k) = \int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{e^{ikr \cos \theta}}{r^2 + \lambda^2} r^2 \sin \theta \ dr \ d\theta \ d\phi \]  \hspace{1cm} (8)
After straightforward integration over \( \phi \) (which yields \( 2\pi \)) and introducing \( u = \cos \theta \) (and with appropriate change in limits of the integral) one obtains:
\[ \tilde{f}(k) = \frac{4\pi}{k} \int_{0}^{\infty} \frac{r \sin(kr)}{r^2 + \lambda^2} dr = \frac{4\pi}{k} \frac{1}{2} \left( \int_{-\infty}^{\infty} \frac{r e^{-ikr}}{r^2 + \lambda^2} dr \right) \]  \hspace{1cm} (9)
The above integration can be done using Jordan’s Lemma due to which the integral is equal to \( 2\pi i \times \) (Residues in the upper half plane). The poles are given by \( z = \pm i\lambda \) of which \( z = i\lambda \) lies in the upper half plane corresponding to which the residue is \( \frac{1}{2} e^{-\lambda} \).
Hence
\[ \tilde{f}(k) = \frac{4\pi^2}{k} e^{-\lambda k} \]  \hspace{1cm} (10)
where \( k = ||k|| \).

[IV]
1. \( \vec{E} \) is odd under Parity. \( \vec{B} \) is even under Parity.
   \( \vec{E} \) is even under Time Reversal, \( \vec{B} \) is odd under Time Reversal.
2. \[
\frac{d\vec{p}}{dt} = \frac{q}{c}(\vec{v} \times \vec{B}), \quad \vec{p} = m\vec{v}
\]  \hspace{1cm} (11)
Let \( \vec{B} = B\hat{k} \). Then
\[
\frac{d\vec{v}}{dt} = \omega(\vec{v} \times \vec{k}), \quad \omega = qB/mc
\]  \hspace{1cm} (12)
Solve component-wise to get
\[ x(t) = -\frac{\alpha}{\omega} \sin \omega t - \frac{\beta}{\omega} \cos \omega t + \delta \]  \hspace{1cm} (13)
\[ y(t) = -\frac{\alpha}{\omega} \cos \omega t + \frac{\beta}{\omega} \sin \omega t + \gamma \]  \hspace{1cm} (14)
\[ x^2 + y^2 = \frac{\alpha^2 + \beta^2}{\omega^2} \]  \hspace{1cm} (15)
is an equation for a circle.
\[ z = at + b \]  \hspace{1cm} (16)
\[ \vec{r}(0) = \vec{r}_0 + \hat{i}(-\beta/\omega) + \hat{j}(-\alpha/\omega) \quad (17) \]
\[ \vec{r}(0 + 2\pi/\omega) = \vec{r}_0 + \hat{i}(-\beta/\omega) + \hat{j}(-\alpha/\omega) + \frac{2\pi a}{\omega} \hat{k} \quad (18) \]

So after one period, the \( z \)-coordinate moves up by \( 2\pi/\omega \). Radius of circle is \( R = \sqrt{\alpha^2 + \beta^2}/\omega \).

Pitch angle:
\[ \tan \theta = \frac{2 |v_z|}{\omega R} \quad (19) \]

[V]
1. Let the gas have \( f \) degrees of freedom. Then

- Isothermal: \( Q = \int \delta Q = \int_{V_i}^{V_f} Nk \frac{T}{V} dV = NkT \ln \left( \frac{V_f}{V_i} \right) \)
- Isochoric: \( Q = \int \delta Q = \frac{f}{2} Nk(T_f - T_i) \)
- Isobaric: \( Q = \int \delta Q = \int_{T_i}^{T_f} (1 + \frac{f}{2}) Nk dT = (1 + \frac{f}{2}) Nk(T_f - T_i) \)

Adding the results, we get \( Q = \oint \delta Q = Nk[T_f - T_i - T_1 \ln \left( \frac{T_f}{T_i} \right)] \)

Equivalently, \( Q = NkT_1 \left[ \frac{T_f}{T_i} - 1 - \ln \left( \frac{T_f}{T_i} \right) \right] \). For \( T_2 \geq T_1 \), \( Q \) is an increasing function of \( T_2/T_1 \). Hence \( Q \) is positive in the cycle.

2. For the reverse cycle, \( Q \) is the same as above, but negative.

3. If \( \delta Q \) were an exact differential, \( Q = 0 \)

4. \( \delta W = -dE = -\frac{f}{2} Nk dT \). When integrated, this gives \( W = -\frac{f}{2} Nk(T_f - T_i) \)

5. \( \delta W = -dE = -\frac{f}{2} Nk dT \). When integrated, this gives \( W = -\frac{f}{2} Nk(T_f - T_i) \)

6. Change in entropy is zero: entropy being a state function, integration over a thermodynamic cycle gives zero.