PG Physics entrance exam 2013: Solutions

[I]

1. $V(\theta) = -mgl\cos\theta$ for $-\pi \le \theta \le \pi$. When $\theta = 0$, the potential is a minimum at $-mgl$. Plot is that of negative cosine function. Lagrangian $L = \frac{1}{2}ml^2\dot\theta^2 + mgl\cos\theta$.

2. $ml^2\ddot\theta + mgl\sin\theta = 0$ or $\ddot\theta = -(g/l)\sin\theta$. Static solution means θ is constant in time or $\dot\theta \equiv 0$. This happens iff $\sin\theta = 0$ which means $\theta = 0, \pm\pi$, in other words the bob is at rest at the bottom or at the top. Former is stable equilibrium and latter unstable.

3. Phase space variables are θ, p_θ. $p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta$. Solve for $\dot\theta$ in terms of p_θ and substitute in $H(\theta, p_\theta) = p_\theta\dot\theta - L(\theta,\dot\theta)$ to find the hamiltonian $H(\theta, p_\theta) = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$.

4. The phase space is an infinite cylinder $-\pi \le \theta \le \pi$ with $\theta = \pm\pi$ being identified and p_θ can be an arbitrary real number.

5. Maximum angle of deflection for any fixed energy in the given range is $\theta_o = \arccos\left[-\frac{E}{mgl}\right]$. Integrate energy equation for $\dot\theta$, $E = \frac{1}{2}ml^2\dot\theta^2 - mgl\cos\theta$ once to get $T = 4\int_0^{\theta_o}\frac{d\theta}{\sqrt{\frac{2}{ml^2}\left[E+mgl\cos\theta\right]}}$. Factor of 4 as energy is even function of $\dot\theta$ and θ, bob goes back and forth and must return to original position with the original velocity, so the trajectory may be broken into 4 parts each taking the same time.

6. As $E \to mgl$, $\theta_o \to \pm\pi$ and $T \to \infty$ since the integrand $\sim 1/\cos(\theta/2)$ has a simple pole at $\theta = \pm\pi$. If bob is raised to initial angles approaching arbitrarily close to $\pm\pi$, then the period of oscillation can be made arbitrarily long.

[II]

1. The possible outomes are $S_z = 2$ and $S_z = 1$. The respective probabilites are

$$\frac{a^2}{a^2 + b^2}\text{and}\frac{\mathrm{b}^2}{\mathrm{a}^2 + \mathrm{b}^2}$$

2. Dimensions of λ and μ are the same. The dimension is $(ML^2)^{-1}$ as we know the dimensions of S^2 and S_x^2 are those of the square of angular momentum $(ML^2/T)^2$ and that of H is energy ML^2/T^2. .

3. The conserved quantities are H, S^2 and$\mathrm{S_x}$.

4. The expectation value of the first term is $6\hbar^2$ as $|\psi>$ is an eigenfunction of S^2 (which are given by $S(S+1)\hbar^2$) with $S = 2$

The expectation value for S_x^2, needed for the second term in the hamiltonian can be found by writing $S_x = (S_+ + S_-)/2$ and using the relations

$$S_+|S,m> = \hbar\sqrt{(S-m)(S+m+1)}|S,m+1>$$

and

$$S_x|S,m> = \hbar\sqrt{S+m)(S-m+1)}|S,m-1>$$

.

The result is

$$< \psi|\mu S_x^2|\psi > = \mu\hbar^2(1 + \frac{3\,|b|^2}{2(|a|^2 + |b|^2}$$

Finally we get

$$< \psi|H|\psi > = 6\,\lambda\,\hbar^2 + \mu\hbar^2(1 + \frac{3\,|b|^2}{2(|a|^2 + |b|^2)})$$

[III]

1. Method: Find Eigen vectors of A, Diagonalize A ($D = S^{-1}\,A\,S$ where S is the matrix formed from Eigen vectors) and take the square root of the diagonal elements of D ($\sqrt{D}$) and transform it back $\sqrt{A} = S\ \sqrt{D}\,S^{-1}$).

   Eigen values: $\lambda = 3, 1$

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

   Finally

$$\sqrt{A} = \begin{pmatrix} 1 & \text{-1+}\sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix}$$

2. Soln:

   The Fourier Transform of the given function is given by

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\,\mathbf{k}.\mathbf{r}}}{r^2 + \lambda^2}dx\,dy\,dz. \quad (1)$$

   Introducing spherical polar coordinates

$$\begin{aligned} x(r,\theta,\phi) &= r\ \cos\phi\sin\theta \\ y(r,\theta,\phi) &= r\ \sin\phi\sin\theta \\ z(r,\theta,\phi) &= r\cos\theta \end{aligned} \quad (2)$$

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\,\mathbf{k}.\mathbf{r}}}{r^2 + \lambda^2}dx\,dy\,dz. \quad (3)$$

Introducing spherical polar coordinates

$$x(r,\theta,\phi) = r\,\cos\phi\sin\theta \tag{4}$$
$$y(r,\theta,\phi) = r\,\sin\phi\sin\theta \tag{5}$$
$$z(r,\theta,\phi) = r\cos\theta \tag{6}$$

immediately leads to

$$dxdydz = r^2 dr\,\sin\theta\, d\theta\, d\phi \tag{7}$$

where $0 \leq \phi < 2\pi, 0 \leq \theta < \pi, r \geq 0$. We also have $\mathbf{k}.\mathbf{r} = kr\cos\theta$ where $k = \|\mathbf{k}\|$. The integral now takes the form

$$\tilde{f}(\mathbf{k}) = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{e^{ikr\cos\theta}}{r^2+\lambda^2} r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{8}$$

After straightforward integration over ϕ (which yields 2π) and introducing $u = \cos\theta$ (and with appropriate change in limits of the integral) one obtains:

$$\tilde{f}(\mathbf{k}) = \frac{4\pi}{k}\int_0^{\infty} \frac{r\sin(kr)}{r^2+\lambda^2} dr = \frac{4\pi}{k}\frac{1}{2}\Im\left(\int_{-\infty}^{\infty}\frac{re^{ikr}}{r^2+\lambda^2}dr\right) \tag{9}$$

The above integration can be done using Jordan's Lemma due to which the integral is equal to $2\pi i\times$ (Residues in the upper half plane). The poles are given by $z = \pm i\lambda$ of which $z = i\lambda$ lies in the upper half plane corresponding to which the residue is $\frac{1}{2}\,e^{-k\lambda}$. Hence

$$\tilde{f}(\mathbf{k}) = \frac{4\pi^2}{k}\,e^{-\lambda k} \tag{10}$$

where $k = |\mathbf{k}|$.

[IV]

1. $\vec{E}$ is odd under Parity. $\vec{B}$ is even under Parity.

$\vec{E}$ is even under Time Reversal, $\vec{B}$ is odd under Time Reversal.

2.

$$\frac{d\vec{p}}{dt} = \frac{q}{c}(\vec{v}\times\vec{B}), \quad \vec{p} = m\vec{v} \tag{11}$$

Let $\vec{B} = B\hat{k}$. Then

$$\frac{d\vec{v}}{dt} = \omega(\vec{v}\times\vec{k}), \quad \omega = qB/mc \tag{12}$$

Solve component-wise to get

$$x(t) = -\frac{\alpha}{\omega}\sin\omega t - \frac{\beta}{\omega}\cos\omega t + \delta \tag{13}$$

$$y(t) = -\frac{\alpha}{\omega}\cos\omega t + \frac{\beta}{\omega}\sin\omega t + \gamma \tag{14}$$

$$x^2 + y^2 = \frac{\alpha^2+\beta^2}{\omega^2} \tag{15}$$

is an equation for a circle.

$$z = at + b \tag{16}$$

$$\vec{r}(0) = \vec{r}_0 + \hat{\imath}(-\beta/\omega) + \hat{\jmath}(-\alpha/\omega) \tag{17}$$

$$\vec{r}(0 + 2\pi/\omega) = \vec{r}_0 + \hat{\imath}(-\beta/\omega) + \hat{\jmath}(-\alpha/\omega) + \frac{2\pi a}{\omega}\hat{k} \tag{18}$$

So after one period, the z-coordinate moves up by $2\pi/\omega$. Radius of circle is $R = \sqrt{\alpha^2 + \beta^2}/\omega$. Pitch angle:

$$\tan\theta = \frac{2\pi \mid v_z \mid}{\omega R} \tag{19}$$

[V]

1. Let the gas have f degrees of freedom. Then

Isothermal: $Q = \int \delta Q = \int_{V_i}^{V_f} Nk\frac{T}{V}dV = NkT\ln(\frac{V_f}{V_i})$

Isochoric: $Q = \int \delta Q = \frac{f}{2}Nk(T_f - T_i)$

Isobaric: $Q = \int \delta Q = \int_{T_i}^{T_f}(1 + \frac{f}{2})NkdT = (1 + \frac{f}{2})Nk(T_f - T_i)$

Adding the results, we get $Q = \oint \delta Q = Nk[T_2 - T_1 - T_1\ln\frac{V_2}{V_1}]$

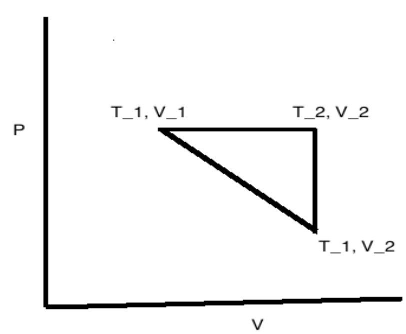
Equivalently, $Q = NkT_1[\frac{T_2}{T_1} - 1 - \ln\frac{T_2}{T_1}]$. For $T_2 \geq T_1$, Q is an increasing function of T_2/T_1. Hence Q is positive in the cycle.

2. For the reverse cycle, Q is the same as above, but negative.

3. If δQ were an exact differential, $Q = 0$

4. $\delta W = -dE = -\frac{f}{2}NkdT$. When integrated, this gives $W = -\frac{f}{2}Nk(T_f - T_i)$

5.



6. Change in entropy is zero: entropy being a state function, integration over a thermodynamic cycle gives zero.