Chennai Mathematical Institute

PhD (Physics) Entrance Examination 2011

1. Given generalised coordinates Q_1, Q_2 and their canonically conjugate momenta P_1, P_2,

   (a) Verify that the following transformation is canonical:

$$X = \sqrt{\frac{1}{m\omega}}(\sqrt{2P_1}\sin Q_1 + P_2), \quad Y = \sqrt{\frac{1}{m\omega}}(\sqrt{2P_1}\cos Q_1 + Q_2)$$

$$P_X = \frac{1}{2}\sqrt{m\omega}(\sqrt{2P_1}\cos Q_1 - Q_2), \quad P_Y = \frac{1}{2}\sqrt{m\omega}(-\sqrt{2P_1}\sin Q_1 + P_2)$$

   (b) Find Hamilton's equations for a particle moving in a plane in a magnetic field described by the vector potential

$$\vec{A} = (-\frac{YB}{2}, \frac{XB}{2}, 0)$$

   in terms of the variables Q_1, Q_2, P_1, P_2 introduced above, using $\omega = eB/mc$.

   Hint: For a particle in a magnetic field $\vec{P} \to \vec{P} - \frac{e}{c}\vec{A}$.

   (c) Solve Hamilton's equations and express them in terms of the original variables X, Y and P_X, P_Y.

   (d) Promote the classical canonical variables to quantum mechanical operators and evaluate $[\hat{\pi}_X, \hat{\pi}_Y]$ where $\hat{\pi}_X = \hat{P}_X - e\hat{A}_X/c$ and $\hat{\pi}_Y = \hat{P}_Y - e\hat{A}_Y/c$.

   (e) Write the quantum mechanical Hamiltonian for a particle moving in a plane in the above magnetic field. By comparing the Hamiltonian and the commutation relation obtained above with those of a quantum mechanical oscillator, show that the energy eigenvalues are

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + (\frac{|\, eB \,|\, \hbar}{mc})(n + \frac{1}{2})$$

2. Suppose a non-relativistic quantum mechanical particle of mass m moving in one space dimension has the initial gaussian wave packet state $\psi(x) = Ae^{-x^2/4a^2}$ at $t = 0$.

   (a) What are the physical dimensions of a and A? Why?

(b) Find the value of A such that the particle can be found somewhere in $[-\infty, \infty]$ with probability one.

(c) Find the expectation value of position $\langle x \rangle$ and standard deviation of position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ in the above initial gaussian wave packet state.

(d) Write down the free particle Schrödinger equation for the time-evolution of the above particle and find its stationary states and their energies.

(e) Suppose a particle in the above initial state $\psi(x, t = 0)$ evolves in time according to the free particle Schrödinger equation. Find the state function $\psi(x, t)$ at time t.

Hint: Express $\psi(x, t = 0)$ in terms of energy eigenstates and evolve them forward in time.

3. Consider propagation of a plane electromagnetic wave with wave vector $\mathbf{k}$ and angular frequency ω in a region containing free electrons of number density n_e .

(a) Write down the equation of motion of an electron in the field of the electromagnetic wave (neglect the effect of magnetic field) and solve for the velocity of the electron as a function of electric field $\mathbf{E}$.

(b) Find the current density induced by the electric field $\mathbf{E}$ (neglect the interaction between electrons) and use the continuity equation to deduce the corresponding charge density in terms of $\mathbf{E}$ and ω.

(c) Write down Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ fields in *(i)* in coordinate space and *(ii)* in $k - \omega$ space.

(d) Show that the wave equation for electric field vector in $k - \omega$ space is of the form $k^2\mathbf{E} = \frac{\omega^2}{c^2}\epsilon\,\mathbf{E}$. Extract ϵ.

(e) Obtain the modified dispersion relation for propagation of the electromagnetic waves. State the condition for electromagnetic waves to propagate indefinitely far in this medium.

4. Consider a one-dimensional system of spin-$\frac{1}{2}$ degrees of freedom localized at the sites of a lattice of spacing a. The Hamiltonian for this system is

$$H = -J_1 \sum_i \sigma_i \sigma_{i+1} + J_2 \sum_i \sigma_i \sigma_{i+2}.$$

The spins interact via (*i*) nearest neighbour ferromagnetic interactions with coupling J_1 and (*ii*) next-to-nearest neighbour anti-ferromagnetic interactions (favouring anti-aligned spins) with coupling J_2.

(a) For a one-dimensional system of three spins with a Hamiltonian of the above form, find the partition function at temperature T.

(b) Use the result of (a) to calculate the entropy and mean energy of the spin configuration.

(c) For nonzero J_2, it is clear that the lowest energy configuration is not necessarily all spins up or down (unlike the Ising model). Consider a 1-dimensional configuration of six spins again with the Hamiltonian of the above form, and find the condition on J_1, J_2 for the ground state to still be all spins up (or down).

Hint: Use the Hamiltonian above to evaluate the energy of various spin configurations.

(d) Now imagine the lattice to be two-dimensional with the ferromagnetic coupling J_1 to be present both in the x and y directions, while the anti-ferromagnetic coupling J_2 is only in the x-direction. Discuss qualitatively the nature of the ground state(s) in this two-dimensional case, if this condition on J_1, J_2 is violated.