CMI PhD (Physics) Entrance Exam: Sample Questions

I. A small particle of mass m slides without friction on the inside of a hemispherical bowl resting on a horizontal table. Using spherical polar coordinates,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

a) Write the Lagrangian for the motion.

b) Determine the generalised momenta p_θ and p_ϕ.

c) Write the Hamiltonian for the motion.

d) Develop the Hamilton's equations of motion.

e) Combine the equations to produce one second order differential equation for θ as a function of time.

f) If $\theta = \theta_0$ and $\dot{\theta} = 0$, independent of time, calculate the velocity (magnitude and direction).

g) If at $t = 0$, $\theta = \theta_0$, and $\dot{\phi} = 0$, calculate the maximum speed at later times.

II. A sphere of radius R has a surface charge density σ.

a) Find the electric potential both inside and outside the sphere.

b) Find the electric field everywhere.

c) Plot a few equipotential surfaces and electric field lines.

The sphere is now set rotating about the z-axis with an angular velocity Ω.

d) Find the vector potential both inside and outside the sphere.

e) Find the magnetic field everywhere.

f) Plot the magnetic field lines.

III. Consider a system composed of a very large number N of distinguishable atoms at rest and mutually noninteracting, each of which has only two (nondegenerate) energy levels: $0, \epsilon > 0$. Let E/N be the mean energy per atom in the limit $N \to \infty$.

a) What is the maximum possible value of E/N if the system is not necessarily in thermodynamic equilibrium?

b) What is the maximum attainable value of E/N if the system is in equilibrium (at positive temperature)?

c) For thermodynamic equilibrium compute the entropy per atom as a function of E/N.

Suppose now that the system is in a thermal radiation field at temperature T. The following three processes take place:

1) Atoms can be promoted from state 1 to state 2 by absorption of a photon according to

$$(\frac{dN_1}{dt})_{abs} = -B_{12}N_1\rho(\nu)$$

2) Atoms can decay from state 2 to state 1 by spontaneous emission according to

$$(\frac{dN_2}{dt})_{spon} = -A_{21}N_2$$

3) Atoms can decay from state 2 to state 1 by stimulated emission according to

$$(\frac{dN_1}{dt})_{sti} = -B_{12}N_1\rho(\nu)$$

The populations N_1 and N_2 are in thermal equilibrium, and the radiation density is

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/T} - 1}$$

d) What is the ratio $\frac{N_2}{N_1}$?

e) Calculate the ratios of coefficients A_{21}/B_{21} and B_{21}/B_{12}.

f) From the ratio of stimulated to spontaneous emission, how does the pump power scale with wavelength when you try to make short wavelength lasers?

IV. Consider a simple harmonic oscillator in one dimension with the usual Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

a) The eigenfunction of the ground state can be written as

$$\psi_0(x) = Ne^{\frac{-\alpha^2 x^2}{2}}$$

Determine the constants N and α.

b) What is the eigenvalue of the ground state?

c) At time $t = 0$, the wavefunction is

$$\psi(x,0) = \sqrt{\frac{1}{3}}\psi_0(x) + \sqrt{\frac{2}{3}}\psi_2(x)$$

where $\psi_n(x)$ is the exact eigenstate of the harmonic oscillator with eigenvalue $\hbar(n+1/2)$.

d) Give $\psi(x,t)$ for $t \geq 0$.

e) What is the parity of this state? Does it change with time?

f) What is the average value of the energy for this state? Does it change with time?