CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
2nd May 2026

Important information and instructions:

(1) Questions in Part A (Questions 1 – 10) will be used for screening. There will be a cut-off for Part A, which will not be more than 20 marks (out of 40).

(2) Each question in Part A has one or more correct answers. Enter your answers to these questions into the computer as instructed. Every question is worth 4 marks. A solution will receive credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(3) Your solutions to the questions in Part B (Questions 11– 20*) will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.)

(4) Answer 6 questions from Part B, on the pages assigned to them, with sufficient justification. Each question is worth 10 marks. Clearly indicate which six questions you would like us to mark in the six boxes on the front sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

(5) The scores in both the sections will be taken into account while making the final decision. You are advised to spend at least 90 minutes on Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least 15 marks from among the starred questions 17*–20*.

(6) Time: 3 hours.

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, and of complex numbers. For a field F and a positive integer n, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}_n(F)$ stands for the set of invertible $n \times n$ matrices over F. The $n \times n$ identity matrix is denoted by I_n; the field will be clear from context. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}^n$ are taken with the euclidean topology, unless otherwise stated.

Part A

(1) Let X be a metric space. A subset $A \subset X$ is said to be *totally bounded* if for every $\epsilon > 0$, there exist $a_1, \ldots, a_n \in A$ such that A is contained in the union of the balls of radius ϵ centered at $a_1, \ldots, a_n$. Pick the correct statement(s) from below.

(A) Every bounded set in $\mathbb{R}^n$ is totally bounded.

(B) Suppose that every sequence in X has a Cauchy subsequence. Then X is totally bounded.

(C) Let $f : X \longrightarrow Y$ be a continuous map of metric spaces. If $A \subset X$ is bounded, then $f(A)$ is bounded in Y.

(D) Let $f : X \longrightarrow Y$ be a uniformly continuous map of metric spaces. If $A \subset X$ is totally bounded, then $f(A)$ is totally bounded in Y.

(2) Let V be a complex vector space. A linear operator $P : V \longrightarrow V$ is called a *projection* if $P^2 = P$. Pick the correct statement(s) from below.

(A) Let P be a projection operator on V. Then the rank of P is equal to the trace of P.
(B) Let P be a projection operator on V. Then 0 is an eigenvalue of P.
(C) Let T be a linear operator on V which commutes with every projection operator. Then every non-zero vector $v \in V$ is an eigenvector of T.
(D) Let T be a linear operator on V which commutes with every projection operator. Then T is a scalar multiple of the identity operator.

(3) Consider the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \frac{xy}{x^2 + xy + y^2}.$$

Pick the correct statement(s) from below.
(A) f is everywhere differentiable and extends differentiably to the whole of $\mathbb{R}^2$.
(B) f is everywhere differentiable and extends continuously but not differentiably to the whole of $\mathbb{R}^2$.
(C) f is everywhere differentiable but does not extend continuously to the whole of $\mathbb{R}^2$.
(D) f is everywhere differentiable and extends to a function g on the whole of $\mathbb{R}^2$ such that both partial derivatives of g exist at $(0,0)$.

(4) Let F be a field of order 2^{12}.
Pick the correct statement(s) from below.
(A) F has exactly 6 subfields, including F.
(B) Every element of F is a square (that is, for every $x \in F$, there exists an element $y \in F$ such that $y^2 = x$).
(C) If K and L are subfields of F of order 8 and 64 respectively, then $K \cup L$ is a subfield of F.
(D) F contains a subfield of order 128.

(5) Let G be a finite group of order p^2q, where $p < q$ are primes. Which of the following statements are *always* true?
(A) If p is odd, then G has a normal Sylow q-subgroup.
(B) If p is odd, then G has a normal and abelian subgroup N such that G/N is abelian.
(C) If p divides $q-1$, then G has at least two Sylow p-subgroups.
(D) G is abelian.

(6) Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers. For each $k \in \mathbb{N}$, define $S_k : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$S_k(x) = \sum_{n=1}^{k} a_n \sin(nx).$$

Let $D = \{x \in \mathbb{R} : \lim_{k\longrightarrow\infty} S_k(x) \text{ exists in } \mathbb{R}\}$, and $f : D \longrightarrow \mathbb{R}$ be defined by $f(x) = \lim_{k\longrightarrow\infty} S_k(x)$.
Pick the correct statement(s) from below.
(A) If $a_n = \frac{1}{\sqrt{n}}$, then $D = \mathbb{R}$.

(B) If $a_n = \frac{1}{n^2}$ and $g : (0,1] \longrightarrow \mathbb{R}$ is defined by $g(x) = S_{N(x)}(x)$, where $N(x)$ is the greatest integer less than or equal to $1/x$, then $\limsup_{x \longrightarrow 0^+} \frac{g(x)}{x} < \infty$.

(C) If $a_n = \frac{1}{n^3}$, then f is Lipschitz continuous on D, i.e.,

$$\sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in D, x \neq y \right\} < \infty.$$

(D) If $a_n = \frac{1}{n}$, then f is continuous on $(0, 2\pi) \cap D$.

(7) Let k be a field and let $R = k[X]/(X^n)$ with $n \geq 2$. Which of the following statements about R-modules are *always* true?

(A) Every finitely generated R-module is a direct sum of cyclic modules.

(B) Every R-module is free.

(C) Every cyclic R-module is isomorphic to R or $R/(\overline{X}^m)$ for some $1 \leq m \leq n$.

(D) Every R-module is cyclic.

(8) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with a radius of convergence exactly equal to 1. Suppose $f(z)$ can be analytically continued to a meromorphic function on the open disk $D = \{z \in \mathbb{C} : |z| < 2\}$, and its only singularity in this region is a pole of order 2 at $z = 1$ with principal part $\frac{c_{-2}}{(z-1)^2} + \frac{c_{-1}}{z-1}$, where $c_{-2}, c_{-1} \neq 0$.

Pick the correct statement(s) from below.

(A) $\lim_{n \longrightarrow \infty} (a_{n+1} - a_n) = 0$.

(B) The series $\sum_{n=0}^{\infty} |a_n| (3/2)^n$ converges.

(C) Let Γ be the contour $|w| = 3/2$ traversed counterclockwise. Then

$$\frac{1}{2\pi i} \oint_{\Gamma} f(w)\, dw \neq 0.$$

(D) Let R_n be the radius of convergence of the Taylor expansion of $f(z)$ centered at $z_n = e^{i\pi/n}$. Then $\lim_{n \longrightarrow \infty} nR_n = \pi$.

(9) Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that $\lim_{n \longrightarrow \infty} a_n = 0$. Pick the correct statement(s) from below.

(A) The limit $\lim_{x \longrightarrow 1^-} \left((1 - x) \sum_{n=0}^{\infty} a_n x^n \right)$ as x approaches 1 from below exists and is equal to 0.

(B) The limit $\lim_{x \longrightarrow 1^+} \left((1 - x) \sum_{n=0}^{\infty} a_n x^n \right)$ as x approaches 1 from above exists and is equal to 0.

(C) The function $f(x) = (1 - x) \sum_{n=0}^{\infty} a_n x^n$ is defined and continuous in the open interval $(0, 1)$.

(D) If the function $f(x) = (1 - x) \sum_{n=0}^{\infty} a_n x^n$ is defined and continuous in the open interval $(0, 1)$, then $|a_n| \leq 1/n$ for all $n \geq 1$.

(10) A sequence $\{C_k\}$ of real 2×2 matrices is said to *converge* to a 2×2 matrix C, written $C_k \longrightarrow C$, if the entries of C_k converge to the corresponding entries of C. Now suppose that A is a real 2×2 matrix such that the sequence $\{A^k\}_{k \geq 1}$ converges to the zero matrix: $A^k \longrightarrow 0$.

Pick the correct statement(s) from below.

(A) The eigenvalues of A are always real.

(B) The eigenvalues λ_j of A must satisfy $|\lambda_j| < 1$, $j = 1, 2$.

(C) The eigenvalues of A must be distinct.
(D) The geometric series $\sum_{k=0}^{\infty} A^k$ converges to $(I - A)^{-1}$.

Part B

(11) Let $k = \mathbb{Z}/p\mathbb{Z}$ where p is a prime. Determine all the ring automorphisms of the polynomial ring $k[X]$.

(12) Let $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$ with only finitely many poles. Assume that $\lim_{z \longrightarrow 0} f(1/z) = 2026$. Let m denote the number of poles of f and let t be the order of the zero of $f(z) - 2026$ at ∞. Determine the number of solutions (counted with multiplicity) to $f(z) = 2026$ in terms of m and t.

Hint: First show that f is a rational function.

(13) Let A be a set (possibly uncountable). For each finite subset F of A, let S_F be a non-empty set of functions from F to $\{0, 1\}$. Assume that the collection $\{S_F \mid F \subseteq A, F \text{ finite}\}$ is consistent, i.e., $\{s|_F \mid s \in S_G\} = S_F$ for every pair $F \subseteq G$ of finite subsets of A. Show that there exists a function $f : A \longrightarrow \{0, 1\}$ such that $f|_F \in S_F$ for every finite subset F of A.

(14) (A) (5 marks) Let $M_2(\mathbb{R}) \cong \mathbb{R}^4$ be the space of real 2×2 matrices with the Euclidean topology. Show that the set of matrices with two distinct eigenvalues (in $\mathbb{C}$) is open in $M_2(\mathbb{R})$.
(B) (5 marks) Let n be a positive integer. Let V be an n-dimensional vector space over a finite field with q elements. Calculate the number of k-dimensional subspaces of V for $1 \leq k \leq n$.

(15) Let V be a finite-dimensional vector space over a field and let $GL(V)$ denote the group of invertible linear transformations $V \longrightarrow V$. Let $V = \oplus_{i=1}^{n} V_i$, where V_i are subspaces of V. Let $G \subset GL(V)$ be a subgroup such that for all $g \in G$, $g(V_i) \subset V_i$ for all $1 \leq i \leq n$. Suppose further that the centraliser of G in $GL(V)$ is contained in the center of $GL(V)$. Show that $n = 1$ and $V_1 = V$.

Hint: Consider the projections $\pi_i : V \longrightarrow V_i$ and show that every element of G commutes with each π_i.

(16) Let F be a finite field with q elements.
(A) (7 marks) Show that every element of F can be written as a sum of two squares. That is, for every $a \in F$, show that there exist $b, c \in F$ such that $a = b^2 + c^2$.
(B) (3 marks) For $a \in F$, let $N(a)$ denote the cardinality of the set $\{(b, c) \in F^2 \mid b^2 + c^2 = a\}$. Show that $\sum_{a \in F} N(a) = q^2$.

(17*) Let $K : [0,1]^2 \longrightarrow \mathbb{R}$ be a continuous function. Let $C([0,1])$ denote the set of all real-valued continuous functions on $[0,1]$.

Let $f \in C([0,1])$. For $x \in [0,1]$, define

$$T(f)(x) := \int_0^1 K(x,y)f(y)\,dy.$$

Also, set $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$.

(A) Show that $T(f) \in C([0,1])$.

(B) Show that there exists a constant $M > 0$, depending only on K, such that

$$\|T(f)\|_\infty \leq M\|f\|_\infty.$$

(C) Let $\mathcal{F} = \{f \in C([0,1]) : \|f\|_\infty \leq 1\}$. Show that the family $\{T(f) : f \in \mathcal{F}\}$ is uniformly bounded and equicontinuous.

(D) Deduce that any sequence (f_n) in $\mathcal{F}$ has a subsequence (f_{n_k}) such that $T(f_{n_k})$ converges uniformly on $[0,1]$.

(18*) Let K be a field and consider the ring of formal power series

$$K[[X]] = \left\{ \sum_{n=0}^{\infty} a_n X^n \ : \ a_n \in K \right\}.$$

(A) Show that an element $f(X) = \sum_{n=0}^{\infty} a_n X^n \in K[[X]]$ is invertible if and only if $a_0 \neq 0$.

(B) Show that every nonzero element $f \in K[[X]]$ can be written uniquely in the form

$$f(X) = X^m u(X),$$

where $m \geq 0$ and $u(X)$ is a unit.

(C) Deduce that every ideal of $K[[X]]$ is of the form (X^n) for some $n \geq 0$. In particular, $K[[X]]$ is a principal ideal domain.

(D) Determine all prime ideals of $K[[X]]$.

(19*) (A) (7 marks) Let G be a finite, non-abelian group satisfying the following property: if H_1, H_2 are distinct proper maximal subgroups of G, then $H_1 \cap H_2$ is trivial. Show that G **cannot** be simple, i.e., show that G has a non-trivial, proper normal subgroup.

Recall that a proper subgroup H of G is called *maximal* if no other proper subgroup contains H.

(B) (3 marks) Let G be a finite, non-abelian group such that every proper subgroup of G is abelian. Show that G **cannot** be simple.

(20*) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuously differentiable function. Suppose that there exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^n$, the following inequality holds:

$$\|f(x) - f(y)\| \geq c\|x - y\|.$$

Prove that f is a global diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$.