CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics

MSc/PhD Entrance Examination

2nd May 2026

Important information and instructions:

(1) Questions in Part A (Questions 1 – 10) will be used for screening. There will be a cut-off for Part A, which will not be more than 20 marks (out of 40).

(2) Each question in Part A has one or more correct answers. Enter your answers to these questions into the computer as instructed. Every question is worth 4 marks. A solution will receive credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(3) Your solutions to the questions in Part B (Questions 11– 20*) will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.)

(4) Answer 6 questions from Part B, on the pages assigned to them, with sufficient justification. Each question is worth 10 marks. Clearly indicate which six questions you would like us to mark in the six boxes on the front sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

(5) The scores in both the sections will be taken into account while making the final decision. You are advised to spend at least 90 minutes on Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least 15 marks from among the starred questions 17*–20*.

(6) Time: 3 hours.

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, and of complex numbers. For a field F and a positive integer n, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}_n(F)$ stands for the set of invertible $n \times n$ matrices over F. The $n \times n$ identity matrix is denoted by I_n; the field will be clear from context. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}^n$ are taken with the euclidean topology, unless otherwise stated.

Part A

(1) Let X be a metric space. A subset $A \subset X$ is said to be *totally bounded* if for every $\epsilon > 0$, there exist $a_1, \dots, a_n \in A$ such that A is contained in the union of the balls of radius ϵ centered at $a_1, \dots, a_n$. Pick the correct statement(s) from below.

(A) Every bounded set in $\mathbb{R}^n$ is totally bounded.

(B) Suppose that every sequence in X has a Cauchy subsequence. Then X is totally bounded.

(C) Let $f : X \longrightarrow Y$ be a continuous map of metric spaces. If $A \subset X$ is bounded, then $f(A)$ is bounded in Y.

(D) Let $f : X \longrightarrow Y$ be a uniformly continuous map of metric spaces. If $A \subset X$ is totally bounded, then $f(A)$ is totally bounded in Y.

Solution:A, B, D. **End Solution**

(A) True. Let A be bounded. Then the closure $\bar{A}$ of A is bounded and closed. So $\bar{A}$ is compact, so totally bounded. Then so is A since it is a subset of a totally bounded set.

(B) True. If A is not totally bounded, we can construct a sequence without a Cauchy subsequence as follows. Choose $\epsilon > 0$ such that finitely many ϵ-balls do not cover X. Choose x_1 such that the ϵ-ball B_1 around x_1 does not cover X. Then choose $x_2 \not\in B_1$. If B_2 is the ϵ-ball around x_2, then $X \neq B_1 \cup B_2$. Choose $x_3 \not\in B_1 \cup B_2$. Note that $d(x_1, x_2) \geq \epsilon, d(x_2, x_3 \geq \epsilon$ and $d(x_2, x_3) \geq \epsilon$. Continuing this way, we construct a sequence $x_1, x_2, x_3, \ldots$ such that $d(x_n, x_m) \geq \epsilon$ for every $m \neq n$. So this sequence does not have a Cauchy subsequence.

(C) False: let $f : (0,1) \longrightarrow \mathbb{R}$, $f(x) = 1/x$.

(D) True. Given $\epsilon > 0$, choose $\delta > 0$ such that $d(a,b) < \delta \Rightarrow d(f(a), f(b)) < \epsilon$. Choose a finite set of points $a_1, \ldots, a_n \in A$ such that δ-balls around a_i cover A. Then ϵ-balls around $f(a_i)$ cover $f(A)$.

(2) Let V be a complex vector space. A linear operator $P : V \longrightarrow V$ is called a *projection* if $P^2 = P$. Pick the correct statement(s) from below.

(A) Let P be a projection operator on V. Then the rank of P is equal to the trace of P.

(B) Let P be a projection operator on V. Then 0 is an eigenvalue of P.

(C) Let T be a linear operator on V which commutes with every projection operator. Then every non-zero vector $v \in V$ is an eigenvector of T.

(D) Let T be a linear operator on V which commutes with every projection operator. Then T is a scalar multiple of the identity operator.

Solution:A, C, D. **End Solution**

(A) is true. Since $P^2 = P$, P is diagonalizable and the only eigenvalues are 0 and 1. So the trace of P is the multiplicity of the eigenvalue 1 and it is also the rank of P.

(B) is false. Take P as the identity operator.

(C) is true: Let $v \in V$ be non-zero. Consider the projection P on to the span of v. Then $Pv = v$. So $Tv = TP(v) = PT(v) = \lambda_v v$. So v is an eigenvector with eigenvalue λ_v.

(D) is true: we claim that every non-zero vector has the same eigenvalue. Let $v, w \in V$ be linearly independent vectors. Then $T(v+w) = Tv+Tw = \lambda_v v + \lambda_w w$. Since $v+w$ is an eigenvector of T with eigenvalue λ_{v+w}, $\lambda_{v+w}(v+w) = \lambda_v v + \lambda_w w$. Hence $\lambda_{v+w} = \lambda_v = \lambda_w$.

(3) Consider the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \frac{xy}{x^2 + xy + y^2}.$$

Pick the correct statement(s) from below.

(A) f is everywhere differentiable and extends differentiably to the whole of $\mathbb{R}^2$.

(B) f is everywhere differentiable and extends continuously but not differentiably to the whole of $\mathbb{R}^2$.

(C) f is everywhere differentiable but does not extend continuously to the whole of $\mathbb{R}^2$.

(D) f is everywhere differentiable and extends to a function g on the whole of $\mathbb{R}^2$ such that both partial derivatives of g exist at $(0,0)$.

Solution:C, D. **End Solution**

f is a ratio of two polynomials and hence differentiable on $\mathbb{R}^2 \setminus \{0,0\}$. Note that $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4 \neq 0$ for $(x,y) \neq (0,0)$. f does not have a limit as $(x,y) \longrightarrow (0,0$ (check the limits along the lines $y = x$ and $y = 0$. So (A), (B) are false and (C) is true. (D) is true if we define $f(0,0) = 0$.

(4) Let F be a field of order 2^{12}.

Pick the correct statement(s) from below.

(A) F has exactly 6 subfields, including F.

(B) Every element of F is a square (that is, for every $x \in F$, there exists an element $y \in F$ such that $y^2 = x$).

(C) If K and L are subfields of F of order 8 and 64 respectively, then $K \cup L$ is a subfield of F.

(D) F contains a subfield of order 128.

Solution:A, B, C. **End Solution**

(A) is true. Subfields of F have order 2^r where r is a divisor of 12. So $r = 1, 2, 3, 4, 6, 12$.

(B) is true. Consider the homomorphism $F \setminus \{0\} \longrightarrow F \setminus \{0\}$ given by $x \mapsto x^2$. The kernel of this map is trivial (since the characteristic of F is 2, $1 = -1$). So this map is injective, hence bijective.

(C) is true. Since $8 = 2^3, 64 = 2^6$ and 3 divides 6. So $K \subset L$ and $K \cup L = L$ is a field.

(D) is false. $128 = 2^7$ and 7 does not divide 12.

(5) Let G be a finite group of order p^2q, where $p < q$ are primes. Which of the following statements are *always* true?

(A) If p is odd, then G has a normal Sylow q-subgroup.

(B) If p is odd, then G has a normal and abelian subgroup N such that G/N is abelian.

(C) If p divides $q - 1$, then G has at least two Sylow p-subgroups.

(D) G is abelian.

Solution:A, B. **End Solution**

Let $|G| = p^2q$, with $p < q$.

(A): we are given that p is odd.

Let n_q be the number of Sylow q-subgroups. By Sylow's theorems:

$$n_q \equiv 1 \pmod{q}, \qquad n_q \mid p^2.$$

Since $n_q \mid p^2$, the possibilities are $1, p, p^2$.

But $p < q$, so p is not congruent to $1 \mod q$. So $n_q \neq p$.

If $n_q = p^2$, then q divides $p^2 - 1 = (p+1)(p-1)$. Since $p < q$ we must have $q = p + 1$. But this forces $p = 2$ contradicting the hypothesis. So $n_q = 1$.

Thus the Sylow q-subgroup is normal.

(B) Let $N \lhd G$ be the Sylow q-subgroup which is normal by the above argument. Then

$$|G/N| = p^2,$$

so G/N is abelian.

(C) Take $p = 2$, $q = 3$ and $G = A_4$, the alternating group on 4 letters.. The Sylow 2-subgroups of A_4 have order 4. A_4 has three elements of order 2 (products of two disjoint 2-cycles) and eight elements of order 3 (3-cycles). So there is only one subgroup of order 4. (C) is false.

(D) Take $p = 2$, $q = 3$ and $G = A_4$. G is not abelian.

Thus the correct answers are: (A), (B).

(6) Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers. For each $k \in \mathbb{N}$, define $S_k : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$S_k(x) = \sum_{n=1}^{k} a_n \sin(nx).$$

Let $D = \{x \in \mathbb{R} : \lim_{k\longrightarrow\infty} S_k(x) \text{ exists in } \mathbb{R}\}$, and $f : D \longrightarrow \mathbb{R}$ be defined by $f(x) = \lim_{k\longrightarrow\infty} S_k(x)$.

Pick the correct statement(s) from below.

(A) If $a_n = \frac{1}{\sqrt{n}}$, then $D = \mathbb{R}$.

(B) If $a_n = \frac{1}{n^2}$ and $g : (0, 1] \longrightarrow \mathbb{R}$ is defined by $g(x) = S_{N(x)}(x)$, where $N(x)$ is the greatest integer less than or equal to $1/x$, then $\limsup_{x\longrightarrow 0^+} \frac{g(x)}{x} < \infty$.

(C) If $a_n = \frac{1}{n^3}$, then f is Lipschitz continuous on D, i.e.,

$$\sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in D, x \neq y\right\} < \infty.$$

(D) If $a_n = \frac{1}{n}$, then f is continuous on $(0, 2\pi) \cap D$.

Solution:A, C, D. **End Solution**

(A) **True.** Note that $S_k(x)$ is the k-th partial sum of the series $\sum_{n=1}^{\infty} a_n \sin(nx)$. If $x \in 2\pi\mathbb{Z}$, then all terms vanish. Assume $x \notin 2\pi\mathbb{Z}$. Using the geometric series

$$\sum_{n=1}^{k} e^{inx} = e^{ix}\frac{1 - e^{ikx}}{1 - e^{ix}},$$

we obtain

$$|S_k(x)| \leq \left|\sum_{n=1}^{k} e^{inx}\right| \leq \frac{2}{|1 - e^{ix}|} = \frac{1}{|\sin(x/2)|}.$$

Since $a_n \downarrow 0$, by Dirichlet's test, the series converges for all $x \in \mathbb{R}$.

(B) **False.** Let us fix $x \in (0, 1]$ and denote $N(x)$ simply by N.

$$\frac{g(x)}{x} = \sum_{n=1}^{N} \frac{\sin(nx)}{nx}\frac{1}{n}.$$

For $1 \leq n \leq N$, we have $0 < nx \leq Nx \leq 1$. Since $t \mapsto \frac{\sin t}{t}$ is continuous on $[0, 1]$, we have $\frac{\sin(nx)}{nx} \geq \min_{t\in[0,1]} \frac{\sin t}{t} := c > 0$. Thus

$$\frac{g(x)}{x} \geq c \sum_{n=1}^{N(x)} \frac{1}{n}.$$

Since the harmonic series diverges, we have $\limsup_{x\longrightarrow 0^+} \frac{g(x)}{x} = \infty$.

(C) **True.** Since $\sum_{n=1}^{\infty} a_n < \infty$, the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly for all $x \in \mathbb{R}$ by the Weierstrass M-test. We note that $\sum_{n=1}^{\infty} na_n < \infty$. By the mean value theorem, for all $x, y \in \mathbb{R}$,

$$|a_n(\sin(nx) - \sin(ny))| \leq a_n \cdot n|x - y|.$$

Summing,

$$|f(x) - f(y)| \leq |x - y| \sum_{n=1}^{\infty} na_n,$$

so f is Lipschitz continuous.

(D) **True.** The argument of part (A) implies $D = \mathbb{R}$. Let $x_0 \in (0, 2\pi)$. There exists $\delta > 0$ such that $x_0 \in I_\delta = [\delta, 2\pi - \delta]$. It is enough to show that f is continuous on I_δ. We apply Dirichlet's test for uniform convergence on I_δ. The sequence $a_n = \frac{1}{n}$ decreases monotonically to 0. By part (A), for all $x \in I_\delta$, the partial sums of $\sin(nx)$ satisfy

$$\left| \sum_{n=1}^{N} \sin(nx) \right| \leq \frac{1}{\sin(\delta/2)}.$$

Thus, the partial sums are uniformly bounded on I_δ. By Dirichlet's test, the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ converges uniformly on I_δ. Because each term $\frac{\sin(nx)}{n}$ is continuous on $\mathbb{R}$, the uniform convergence ensures that the limit function $f(x)$ is continuous on I_δ.

(7) Let k be a field and let $R = k[X]/(X^n)$ with $n \geq 2$. Which of the following statements about R-modules are *always* true?

(A) Every finitely generated R-module is a direct sum of cyclic modules.
(B) Every R-module is free.
(C) Every cyclic R-module is isomorphic to R or $R/(\overline{X}^m)$ for some $1 \leq m \leq n$.
(D) Every R-module is cyclic.

Solution:A, C. **End Solution**

Write $\overline{X}$ for the image of X in R. Then $\overline{X}^n = 0$, so multiplication by $\overline{X}$ defines a nilpotent operator on any finitely generated R-module M.

Every finitely generated R-module M is a finite-dimensional vector space over k, equipped with a nilpotent linear operator given by multiplication by $\overline{X}$. By linear algebra, such a module decomposes into a direct sum of Jordan blocks.

Each Jordan block corresponds to a module of the form

$$R/(\overline{X}^m).$$

From the above, every finitely generated R-module is a direct sum of cyclic modules of the form $R/(\overline{X}^m)$. (A) is true.

(B) is false. *Counterexample:* Consider the ideal $(\overline{X})$ of R which has a natural structure of an R-module. It is not free: it were free, it would be isomorphic to R, but it contains torsion elements since

$$\overline{X} \cdot \overline{X}^{n-1} = 0.$$

Thus it is not free.

Let $M = Rf$ be a cyclic module. Then $M \cong R/\operatorname{Ann}(f)$. Since every ideal of R is of the form $(\overline{X}^m)$, we get

$$M \cong R \quad \text{or} \quad R/(\overline{X}^m)$$

for some $1 \leq m \leq n$. Thus, (C) is true.

(D) is false: for example, the R-module $R/(\overline{X}) \oplus R/(\overline{X})$ is not cyclic.

(8) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with a radius of convergence exactly equal to 1. Suppose $f(z)$ can be analytically continued to a meromorphic function on the open disk $D = \{z \in \mathbb{C} : |z| < 2\}$, and its only singularity in this region is a pole of order 2 at $z = 1$ with principal part $\frac{c_{-2}}{(z-1)^2} + \frac{c_{-1}}{z-1}$, where $c_{-2}, c_{-1} \neq 0$.

Pick the correct statement(s) from below.

(A) $\lim_{n \longrightarrow \infty}(a_{n+1} - a_n) = 0$.

(B) The series $\sum_{n=0}^{\infty} |a_n|(3/2)^n$ converges.

(C) Let Γ be the contour $|w| = 3/2$ traversed counterclockwise. Then

$$\frac{1}{2\pi i} \oint_{\Gamma} f(w)\, dw \neq 0.$$

(D) Let R_n be the radius of convergence of the Taylor expansion of $f(z)$ centered at $z_n = e^{i\pi/n}$. Then $\lim_{n \longrightarrow \infty} nR_n = \pi$.

Solution:C, D. **End Solution**

(A) **False.** We can write $f(z) = h(z) + \frac{c_{-2}}{(z-1)^2} + \frac{c_{-1}}{z-1}$, where $h(z) = \sum_{n=0}^{\infty} b_n z^n$ is holomorphic on $|z| < 2$. Using the power series for $(z-1)^{-1}$ and $(z-1)^{-2}$, we obtain $a_n = c_{-2}(n+1) - c_{-1} + b_n$. Since the radius of convergence of $h(z)$ is at least 2, $b_n \longrightarrow 0$. Thus, $\lim_{n \longrightarrow \infty}(a_{n+1} - a_n) = c_{-2} \neq 0$.

(B) **False.** The radius of convergence of the series $\sum_{n=0}^{\infty} |a_n| z^n$ is equal to the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$, which is 1. So $\sum_{n=0}^{\infty} |a_n|(3/2)^n$ cannot converge.

(C) **True.** The integral is equal to $c_{-1} \neq 0$ by Cauchy's Residue Theorem.

(D) **True.** The distance from $z_n = e^{i\pi/n}$ to the nearest singularity $z = 1$ is $R_n = |e^{i\pi/n} - 1| = 2\sin(\frac{\pi}{2n})$. Therefore, $\lim_{n \longrightarrow \infty} nR_n = \pi$.

(9) Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that $\lim_{n \longrightarrow \infty} a_n = 0$. Pick the correct statement(s) from below.

(A) The limit $\lim_{x \longrightarrow 1^-} \left((1-x) \sum_{n=0}^{\infty} a_n x^n\right)$ as x approaches 1 from below exists and is equal to 0.

(B) The limit $\lim_{x \longrightarrow 1^+} \left((1-x) \sum_{n=0}^{\infty} a_n x^n\right)$ as x approaches 1 from above exists and is equal to 0.

(C) The function $f(x) = (1-x) \sum_{n=0}^{\infty} a_n x^n$ is defined and continuous in the open interval $(0, 1)$.

(D) If the function $f(x) = (1-x) \sum_{n=0}^{\infty} a_n x^n$ is defined and continuous in the open interval $(0, 1)$, then $|a_n| \leq 1/n$ for all $n \geq 1$.

Solution:A, C. **End Solution**

Let $\epsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon/2$. For any $M > N$ and $0 < x < 1$, we have the estimate

$$\begin{aligned}|(1-x)\sum_{n=0}^{M} a_n x^n| &\leq |(1-x)\sum_{n=0}^{N} a_n x^n| + |(1-x)\sum_{n=N+1}^{M} a_n x^n| \\ &\leq |(1-x)\sum_{n=0}^{N} a_n x^n| + |(1-x)\sum_{n=N+1}^{M} \frac{\epsilon}{2} x^n| \\ &\leq |(1-x)\sum_{n=0}^{N} a_n x^n| + \epsilon/2.\end{aligned}$$

Clearly, there exists $\delta > 0$ such that $|(1-x)\sum_{n=0}^{N} a_n x^n| \leq \epsilon/2$ for $(1-x) < \delta$. So one sees that (A) holds. Statement (C) holds because $\sum_{n=0}^{\infty} a_n x^n$ is a power series (centered at 0) that converges in $(0,1)$ by a similar estimate to the one above.

(B) and (D) are false for $a_n = 1/\sqrt{n}$.

(10) A sequence $\{C_k\}$ of real 2×2 matrices is said to *converge* to a 2×2 matrix C, written $C_k \longrightarrow C$, if the entries of C_k converge to the corresponding entries of C. Now suppose that A is a real 2×2 matrix such that the sequence $\{A^k\}_{k \geq 1}$ converges to the zero matrix: $A^k \longrightarrow 0$.

Pick the correct statement(s) from below.

(A) The eigenvalues of A are always real.
(B) The eigenvalues λ_j of A must satisfy $|\lambda_j| < 1$, $j = 1, 2$.
(C) The eigenvalues of A must be distinct.
(D) The geometric series $\sum_{k=0}^{\infty} A^k$ converges to $(I-A)^{-1}$.

Solution:B, D. **End Solution**

(A) is false: take $A = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

(B) is true: let $\lambda \in \mathbb{C}$ be an eigenvalue of A and let v be the corresponding eigenvector. So $A^k v = \lambda^k v$ converges to the zero vector as $k \longrightarrow \infty$. So $\lambda^k \longrightarrow 0$.

(C) is false: take $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

(D) is true: let S_k denote the sequence of partial sums: $S_k = I + A + \cdots + A^k$. Then $(I-A)S_k = I - A^{k+1}$. So $(I-A)S_k$ converge to I and hence the series $\sum_k (I-A)A^k$ converges to I. This shows that the series $\sum_k A^k$ also converges, say to B. Then $(I-A)B = I$ and hence $B = (I-A)^{-1}$.

Part B

(11) Let $k = \mathbb{Z}/p\mathbb{Z}$ where p is a prime. Determine all the ring automorphisms of the polynomial ring $k[X]$.

Solution:

Step 1: Action on constants.

Since φ is a ring automorphism, it restricts to an automorphism of the subring k. But $k = \mathbb{Z}/p\mathbb{Z}$ has no nontrivial automorphisms, so

$$\varphi(a) = a \quad \text{for all } a \in k.$$

Step 2: Determination by the image of X.

The automorphism φ is completely determined by $\varphi(X)$, since every polynomial is a polynomial in X with coefficients in k.

Let

$$\varphi(X) = f(X) \in k[X].$$

Step 3: Degree considerations.

Since φ is an automorphism, it must be surjective. This implies that

$$\deg(\varphi(X)) = 1.$$

Thus $f(X)$ must be a linear polynomial:

$$f(X) = aX + b, \quad a, b \in k.$$

Step 4: Invertibility condition.

For φ to be invertible, we must have $a \neq 0$. Indeed, if $a = 0$, then $\varphi(X) = b$ is constant, and φ is not injective.

Conversely, if $a \neq 0$, define

$$\psi(X) = a^{-1}(X - b).$$

Then one checks that ψ is the inverse of φ, so φ is an automorphism.

Step 5: Conclusion.

All ring automorphisms of $k[X]$ are of the form

$$\varphi(X) = aX + b, \quad a \in k^{\times},\ b \in k.$$

Thus,

$$\mathrm{Aut}(k[X]) \cong \{X \mapsto aX + b : a \in k^{\times},\, b \in k\},$$

which is the affine group over k.

End Solution

(12) Let $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$ with only finitely many poles. Assume that $\lim\limits_{z \longrightarrow 0} f(1/z) = 2026$. Let m denote the number of poles of f and let t be the order of the zero of $f(z) - 2026$ at ∞. Determine the number of solutions (counted with multiplicity) to $f(z) = 2026$ in terms of m and t.

Hint: First show that f is a rational function.

Solution:

Step 1: f is a rational function. *Proof:* Let $a_1, \ldots, a_m$ be the distinct poles of f. For each $1 \leq i \leq m$, write

$$f(z) = \sum_{k=-N_i}^{-1} \frac{b_{i,k}}{z - a_i} + f_i(z)$$

where $f_i(z)$ is holomorphic in a neighbourhood of a_i. Write

$$P(z) = \sum_{i=1}^{m} \sum_{k=-N_i}^{-1} \frac{b_{i,k}}{z - a_i}.$$

Then $f(z) - P(z)$ does not have poles on C. Further,

$$\lim_{z \longrightarrow 0} [f(1/z) - P(1/z)] = c,$$

so $f(z)$ has a removable singularity at ∞. Hence f is a rational function.

Step 2: Write

$$f(z) = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials and Q is monic. Therefore $f(z) = 2026$ if and only if $P(z) - 2026Q(z) = 0$. Let $n = \deg P, m = \deg Q$. Since $\lim\limits_{z \longrightarrow 0} f(1/z) = 2026$ is finite, $n = m$.

Step 3: Let t be the order of the zero of $f(z) - 2026$ at ∞. Hence $\deg(P(z) - 2026Q(z)) = m - t$. Therefore there are exactly $m - t$ solutions (counted with multiplicity) for the equation $f(z) = 2026$. $\square$

End Solution

(13) Let A be a set (possibly uncountable). For each finite subset F of A, let S_F be a non-empty set of functions from F to $\{0, 1\}$. Assume that the collection $\{S_F \mid F \subseteq A, F \text{ finite}\}$ is consistent, i.e., $\{s|_F \mid s \in S_G\} = S_F$ for every pair $F \subseteq G$ of finite subsets of A. Show that there exists a function $f : A \longrightarrow \{0, 1\}$ such that $f|_F \in S_F$ for every finite subset F of A.

Solution:Step 1: For each $\alpha \in A$, let

$$X_\alpha = \{0, 1\}$$

with the discrete topology. Let

$$X = \prod_{\alpha \in A} X_\alpha.$$

with the product topology. It is compact by Tychonoff's theorem.

Step 2: For each finite $F \subset A$, define

$$C_F = \{f \in X \mid f|_F \in S_F\}.$$

Each C_F is closed and non-empty.

Step 3: Let $F_1, \dots, F_m$ be finite subsets of A and let $F = \bigcup\limits_{i=1}^{m} F_i$. Then $\bigcap\limits_{i=1}^{m} C_{F_i} = C_F \neq \varnothing$. Since X is compact, there exists

$$f \in \bigcap_{\substack{F \subseteq A \\ F \text{ finite}}} C_F.$$

Then $f|_F \in S_F$ for every finite subset F of A. $\square$

End Solution

(14) (A) (5 marks) Let $M_2(\mathbb{R}) \cong \mathbb{R}^4$ be the space of real 2×2 matrices with the Euclidean topology. Show that the set of matrices with two distinct eigenvalues (in $\mathbb{C}$) is open in $M_2(\mathbb{R})$.

(B) (5 marks) Let n be a positive integer. Let V be an n-dimensional vector space over a finite field with q elements. Calculate the number of k-dimensional subspaces of V for $1 \leq k \leq n$.

Solution:

(A) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial of A is

$$p_A(t) = t^2 - (a+d)t + (ad - bc).$$

The eigenvalues are distinct if and only if the discriminant of this quadratic is nonzero:

$$\Delta(A) = (a+d)^2 - 4(ad - bc).$$

Simplifying,

$$\Delta(A) = (a-d)^2 + 4bc.$$

Thus A has two distinct eigenvalues if and only if

$$\Delta(A) \neq 0.$$

Now observe that $\Delta(A)$ is a polynomial in the entries a, b, c, d, hence defines a continuous function

$$\Delta : M_2(\mathbb{R}) \longrightarrow \mathbb{R}.$$

Therefore, the set of matrices with distinct eigenvalues is

$$\{A \in M_2(\mathbb{R}) : \Delta(A) \neq 0\} = \Delta^{-1}(\mathbb{R} \setminus \{0\}).$$

Since $\mathbb{R} \setminus \{0\}$ is open and Δ is continuous, this set is open in $M_2(\mathbb{R})$.

(B) To count the number of k-dimensional subspaces, first count the number of ordered bases $\{v_1, \ldots, v_k\}$ of cardinality k. Since V has q^n elements, there are $q^n - 1$ choices for v_1, $q^n - q$ choices for v_2 and so on. So the number of ordered bases of cardinality k is $(q^n - 1)(q^n - q) \cdot (q^n - q^{k-1})$.

Now a given k-dimensional subspace of V has many ordered bases. In fact, since its dimension is k, the number of ordered bases of W is precisely $(q^k - 1)(q^k - q) \cdot (q^k - q^{k-1})$.

So the number of k-dimensional subspaces of V is $\frac{(q^n-1)(q^n-q)\cdot(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdot(q^k-q^{k-1})}$.

End Solution

(15) Let V be a finite-dimensional vector space over a field and let $GL(V)$ denote the group of invertible linear transformations $V \longrightarrow V$. Let $V = \oplus_{i=1}^n V_i$, where V_i are subspaces of V. Let $G \subset GL(V)$ be a subgroup such that for all $g \in G$, $g(V_i) \subset V_i$ for all $1 \leq i \leq n$. Suppose further that the centraliser of G in $GL(V)$ is contained in the center of $GL(V)$. Show that $n = 1$ and $V_1 = V$.

Hint: Consider the projections $\pi_i : V \longrightarrow V_i$ and show that every element of G commutes with each π_i.

Solution: For $v \in V$, let $v = v_1 + \cdots + v_n$ be the unique decomposition of v determined by $V = \oplus_{i=1}^n V_i$. Then $\pi_i(v) = v_i$.

Let $g \in G$. Then $g\pi_i(v) = g(v_i) \in V_i$ by hypothesis.

On the other hand, since $g(v_i) \in V_i$ for each i, $g(v) = g(v_1) + \cdots + g(v_n)$ is the unique decomposition of $g(v)$ associated to $V = \oplus_{i=1}^n V_i$ and $g(v_i)$ is the i-th projection of $g(v)$. So $\pi_i g(v) = \pi_i(g(v_1) + \cdots + g(v_n)) = g(v_i)$.

Hence each π_i is in the centralizer of G and so in the center of $GL(V)$. So each π_i is a scalar multiple of the identity map.

Let $\pi_i = c_i I$ for some c_i in the field. Since $\pi_i^2 = \pi_i$, we get $c_i^2 = c_i$. Then $c_i = 1$ or $c_i = 0$. In the latter case, $\pi_i = 0$, so $V_i = 0$. If $c_i = 1$, then $\pi_i = I$. So there is only one non-zero V_i and it is equal to V.

End Solution

(16) Let F be a finite field with q elements.

(A) (7 marks) Show that every element of F can be written as a sum of two squares. That is, for every $a \in F$, show that there exist $b, c \in F$ such that $a = b^2 + c^2$.

(B) (3 marks) For $a \in F$, let $N(a)$ denote the cardinality of the set $\{(b,c) \in F^2 \mid b^2 + c^2 = a\}$. Show that $\sum\limits_{a \in F} N(a) = q^2$.

Solution:

(A) First assume that the characteristic of F is 2. Then in fact every element of F is a square: the map $F \longrightarrow F, a \mapsto a^2$ is a bijection of F. So for all $a \in F$, there exists $b \in F$ such that $a = b^2$.

Now let the characteristic of F be odd. Let $F^\times$ denote the multiplicative group of non-zero elements of F. The homomorphism $F^\times \longrightarrow F^\times$, $a \mapsto a^2$ has kernel of order 2. So exactly half the elements of $F^\times$ are squares and F has exactly $\frac{q-1}{2} + 1 = \frac{q+1}{2}$ squares. Let $a \in F$. The set $\{a - b^2 \mid b \in F\}$ also has cardinality equal to $\frac{q+1}{2}$. The two sets $\{b^2 \mid b \in F\}$ and $\{a - b^2 \mid b \in F\}$ cannot be disjoint, since their cardinalities add up to $q + 1 > |F|$. So there exists $c \in F$ such that $a = b^2 + c^2$.

(B) $\sum\limits_{a \in F} N(a) = \sum\limits_{a \in F} \sum\limits_{\substack{(b,c) \in F^2 \\ b^2+c^2=a}} 1 = \sum\limits_{(b,c) \in F^2} 1 = |F^2| = q^2$.

End Solution

(17*) Let $K : [0,1]^2 \longrightarrow \mathbb{R}$ be a continuous function. Let $C([0,1])$ denote the set of all real-valued continuous functions on $[0,1]$.

Let $f \in C([0,1])$. For $x \in [0,1]$, define

$$T(f)(x) := \int_0^1 K(x,y) f(y)\, dy.$$

Also, set $||f||_\infty := \sup\limits_{x \in [0,1]} |f(x)|$.

(A) Show that $T(f) \in C([0,1])$.

(B) Show that there exists a constant $M > 0$, depending only on K, such that

$$\|T(f)\|_\infty \leq M \|f\|_\infty.$$

(C) Let $\mathcal{F} = \{f \in C([0,1]) : \|f\|_\infty \leq 1\}$. Show that the family $\{T(f) : f \in \mathcal{F}\}$ is uniformly bounded and equicontinuous.

(D) Deduce that any sequence (f_n) in $\mathcal{F}$ has a subsequence (f_{n_k}) such that $T(f_{n_k})$ converges uniformly on $[0,1]$.

Solution:

(A) Let $x_n \longrightarrow x$. Then

$$|T(f)(x_n) - T(f)(x)| = \left| \int_0^1 (K(x_n, y) - K(x,y)) f(y)\, dy \right|.$$

Since $|f(y)| \leq \|f\|_\infty$,

$$|T(f)(x_n) - T(f)(x)| \leq \|f\|_\infty \int_0^1 |K(x_n, y) - K(x, y)|\, dy.$$

Because K is continuous on the compact set $[0,1]^2$, it is uniformly continuous, so

$$\sup_y |K(x_n, y) - K(x, y)| \longrightarrow 0.$$

Hence the integral tends to 0, and $T(f)$ is continuous.

(B) We estimate:

$$|T(f)(x)| \leq \int_0^1 |K(x,y)|\,|f(y)|\, dy \leq \|f\|_\infty \int_0^1 |K(x,y)|\, dy.$$

Taking supremum over x,

$$\|T(f)\|_\infty \leq \|f\|_\infty \cdot \sup_{x\in[0,1]} \int_0^1 |K(x,y)|\, dy.$$

Since K is continuous on a compact set, the supremum is finite. Call it M. Then

$$\|T(f)\|_\infty \leq M \|f\|_\infty.$$

(C) *Uniform boundedness:* If $\|f\|_\infty \leq 1$, then by part (b),

$$\|T(f)\|_\infty \leq M,$$

so the family is uniformly bounded.

Equicontinuity: For $x_1, x_2 \in [0,1]$,

$$|T(f)(x_1) - T(f)(x_2)| \leq \int_0^1 |K(x_1,y) - K(x_2,y)|\,|f(y)|\, dy \leq \int_0^1 |K(x_1,y) - K(x_2,y)|\, dy.$$

The right-hand side does not depend on f, and since K is uniformly continuous,

$$\sup_y |K(x_1,y) - K(x_2,y)| \longrightarrow 0 \quad \text{as } x_1 \longrightarrow x_2.$$

Hence the family is equicontinuous.

(D) Let (f_n) be a sequence in $\mathcal{F}$. Then $(T(f_n))$ is uniformly bounded and equicontinuous by part (c).

By the Arzelà–Ascoli theorem, there exists a subsequence $(T(f_{n_k}))$ which converges uniformly on $[0,1]$.

End Solution

(18*) Let K be a field and consider the ring of formal power series

$$K[[X]] = \left\{ \sum_{n=0}^{\infty} a_n X^n \ :\ a_n \in K \right\}.$$

(A) Show that an element $f(X) = \sum_{n=0}^{\infty} a_n X^n \in K[[X]]$ is invertible if and only if $a_0 \neq 0$.

(B) Show that every nonzero element $f \in K[[X]]$ can be written uniquely in the form

$$f(X) = X^m u(X),$$

where $m \geq 0$ and $u(X)$ is a unit.

(C) Deduce that every ideal of $K[[X]]$ is of the form (X^n) for some $n \geq 0$. In particular, $K[[X]]$ is a principal ideal domain.

(D) Determine all prime ideals of $K[[X]]$.

Solution:

(A) ($\Rightarrow$) Suppose $f(X)$ is invertible, so there exists $g(X) = \sum_{n=0}^{\infty} b_n X^n$ such that $f(X)g(X) = 1$. Looking at the constant term of the product,

$$a_0 b_0 = 1,$$

so $a_0 \neq 0$.

($\Leftarrow$) Suppose $a_0 \neq 0$. We construct $g(X) = \sum b_n X^n$ such that $f(X)g(X) = 1$. Equating coefficients gives:

$$a_0 b_0 = 1 \quad \Rightarrow \quad b_0 = a_0^{-1},$$

and for $n \geq 1$,

$$\sum_{k=0}^{n} a_k b_{n-k} = 0.$$

This determines b_n recursively:

$$b_n = -a_0^{-1} \sum_{k=1}^{n} a_k b_{n-k}.$$

Thus $g(X)$ exists, so $f(X)$ is invertible.

(B) Let $f(X) = \sum_{n=0}^{\infty} a_n X^n \neq 0$, and let m be the smallest index such that $a_m \neq 0$. Then

$$f(X) = X^m \left(a_m + a_{m+1}X + a_{m+2}X^2 + \cdots\right).$$

Set

$$u(X) = a_m + a_{m+1}X + \cdots .$$

Then $u(0) = a_m \neq 0$, so by part (a), $u(X)$ is a unit. Uniqueness follows from the uniqueness of the smallest index m.

(C) Let $I \subset K[[X]]$ be a nonzero ideal. Choose a nonzero element $f \in I$ with minimal m as in part (b), so $f = X^m u$ with u a unit. Then $X^m = fu^{-1} \in I$, so $(X^m) \subset I$.

Conversely, if $g \in I$, write $g = X^n v$ with v a unit. By minimality of m, we must have $n \geq m$, so $g \in (X^m)$. Hence $I = (X^m)$.

Thus every ideal is principal.

(D) From part (c), every ideal is of the form (X^n).

An ideal (X^n) is prime if and only if whenever $fg \in (X^n)$, then $f \in (X^n)$ or $g \in (X^n)$.

If $n \geq 2$, then $X \cdot X^{n-1} = X^n \in (X^n)$ only when $n \leq 2$, but neither factor lies in (X^n) for $n \geq 2$. Hence (X^n) is not prime for $n \geq 2$. Note that (X) is a prime ideal since $K[[X]]/(X) \cong K$ is an integral domain.

Thus the only nonzero prime ideal is (X). The zero ideal (0) is also prime since $K[[X]]$ is an integral domain.

Therefore, the prime ideals are: $(0) \quad \text{and} \quad (X)$.

End Solution

(19^*) (A) (7 marks) Let G be a finite, non-abelian group satisfying the following property: if H_1, H_2 are distinct proper maximal subgroups of G, then $H_1 \cap H_2$ is trivial. Show that G **cannot** be simple, i.e., show that G has a non-trivial, proper normal subgroup.

Recall that a proper subgroup H of G is called *maximal* if no other proper subgroup contains H.

(B) (3 marks) Let G be a finite, non-abelian group such that every proper subgroup of G is abelian. Show that G **cannot** be simple.

Solution:

(A) Assume for a contradiction that G is simple. Let H be a maximal subgroup of G. Then H cannot be trivial (since G is not abelian). Further the normalizer $N(H)$ of H cannot be G since G is simple. So maximality forces $N(H) = H$. Suppose that the order of H is m and the order of G is n. The number of conjugates of H in G is n/m and each of them is also maximal, since H is maximal. Let S denote the union of all the conjugates of H in G. By hypothesis, the intersection of any two of these conjugates is trivial. So S has exactly $\frac{(m-1)n}{m} + 1 = n - \frac{n}{m} + 1$ elements.

Since H is proper, $|S| = n - \frac{n}{m} + 1 < n$. Since $m \geq 2$, we also have $|S| = n - \frac{n}{m} + 1 \geq \frac{n}{2} + 1$.

Now let $x \in G \setminus S$. Then x must lie in a proper maximal subgroup of G, say H'. By the same argument as above the union S' of all the conjugates of H' also has at least $\frac{n}{2} + 1$ elements. Note that a conjugate of H and a conjugate of H' intersect trivially. So $S \cap S' = \{e\}$. But then $S \cup S'$ has at least $n + 1$ elements which is a contradiction.

(B) Assume that G is simple for a contradiction. We show that the intersection of any two maximal proper subgroups of G is trivial. Then we will be done by part (A). Let H_1, H_2 be distinct maximal subgroups of G. By hypothesis, H_1, H_2 are abelian. Let $x \in H_1 \cap H_2$. Then x commutes with every element in H_1 and in H_2 and hence with every element in $H_1 H_2 = G$. So x is in the center of G. If G is simple then its center is trivial (since it is not abelian). So $x = e$. **End Solution**

(20*) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuously differentiable function. Suppose that there exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^n$, the following inequality holds:

$$\|f(x) - f(y)\| \geq c\|x - y\|.$$

Prove that f is a global diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Solution: To prove that f is a global diffeomorphism, we must show that f is bijective and that both f and its inverse f^{-1} are continuously differentiable. Since $f \in C^1(\mathbb{R}^n)$, the Inverse Function Theorem implies that if f is bijective and its derivative $Df(x)$ is invertible at every $x \in \mathbb{R}^n$, then f^{-1} is also C^1, making f a diffeomorphism. We divide the proof into four steps.

Step 1: Injectivity. Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R}^n$. Then $0 = \|f(x) - f(y)\| \geq c\|x - y\|$ which implies $x = y$. Therefore, f is globally injective.

Step 2: Local Invertibility. Fix an arbitrary point $x \in \mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be a non-zero vector.

$$Df(x)v = \lim_{t \longrightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Hence

$$\|Df(x)v\| = \lim_{t \longrightarrow 0} \frac{\|f(x + tv) - f(x)\|}{|t|} \geq \lim_{t \longrightarrow 0} \frac{c\|(x + tv) - x\|}{|t|} = c\|v\|.$$

Since $c > 0$, $\|Df(x)v\| > 0$ for all $v \neq 0$. This implies that the kernel of $Df(x)$ is trivial. Thus, $Df(x)$ is invertible for all $x \in \mathbb{R}^n$.

By the Inverse Function Theorem, f is a local diffeomorphism everywhere. Consequently, f is an open map; so $f(\mathbb{R}^n)$ is an open set in $\mathbb{R}^n$.
Step 3: Closed Image. We now show that $f(\mathbb{R}^n)$ is a closed set. Let $\{y_k\}$ be a sequence in the image $f(\mathbb{R}^n)$ that converges to some limit $y \in \mathbb{R}^n$. We need to show that $y \in f(\mathbb{R}^n)$.

Since $y_k \in f(\mathbb{R}^n)$, there exist $x_k \in \mathbb{R}^n$ such that $f(x_k) = y_k$. Since $\{y_k\}$ is a Cauchy sequence and

$$\|x_j - x_k\| \leq \frac{1}{c}\|f(x_j) - f(x_k)\| = \frac{1}{c}\|y_j - y_k\|,$$

it follows that $\{x_k\}$ is a Cauchy sequence in $\mathbb{R}^n$. Hence the sequence $\{x_k\}$ converges to some $x \in \mathbb{R}^n$. Since f is continuously differentiable, it is continuous. Therefore $f(x) = \lim_{k \longrightarrow \infty} f(x_k) = y$. Thus, $y = f(x)$.
Step 4: Surjectivity and Conclusion. Since $f(\mathbb{R}^n)$ is both open and closed in $\mathbb{R}^n$, $f(\mathbb{R}^n) = \mathbb{R}^n$. Since f is a bijective map of class C^1 with a non-singular Jacobian everywhere, it is a global diffeomorphism.

End Solution