CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
24th May 2025

Important information and instructions:

(1) Questions in Part A (Questions 1 – 10) will be used for screening. There will be a cut-off for Part A, which will not be more than 20 marks (out of 40).

(2) Each question in Part A has one or more correct answers. Enter your answers to these questions into the computer as instructed. Every question is worth 4 marks. A solution will receive credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(3) Your solutions to the questions in Part B (Questions 11– 20*) will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.)

(4) Answer 6 questions from Part B, on the pages assigned to them, with sufficient justification. Each question is worth 10 marks. Clearly indicate which six questions you would like us to mark in the six boxes on the front sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

(5) The scores in both the sections will be taken into account while making the final decision. You are advised to spend at least 90 minutes on Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least 15 marks from among the starred questions 17*–20*.

(6) Time: 3 hours.

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, and of complex numbers. For a prime power q, $\mathbb{F}_q$ is the field with q elements. For a field F and a positive integer n, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}_n(F)$ stands for the set of invertible $n \times n$ matrices over F. The $n \times n$ identity matrix is denoted by I_n; the field will be clear from context. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}^n$ are taken with the euclidean topology, unless otherwise stated.

Part A

(1) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $T \neq 0$ and $T^4 = 0$. Pick the correct statement(s) from below.
  (A) $T^3 = 0$.
  (B) $\mathrm{Image}(T) \neq \mathrm{Image}(T^2)$.
  (C) $\mathrm{rank}(T^2) \leq 1$.
  (D) $\mathrm{rank}(T) = 2$.

(2) Let $W = \{(a, b, c, d) \in \mathbb{R}^4 \mid 3a - b + 6c = 0\}$ and $T : \mathbb{R}^4 \longrightarrow W$ be a linear map with $T^2 = T$. Suppose T is onto. Pick the correct statement(s) from below.
  (A) $T(u + v) = T(u) + v$ for all $u \in \mathbb{R}^4$, $v \in W$.
  (B) $\ker(T - I)$ contains three linearly independent vectors.
  (C) $(1, 3, 0, 2) \in \ker(T)$.
  (D) If $v_1, v_2 \in \ker(T)$ are nonzero, then $v_1 = cv_2$ for some $c \in \mathbb{R}$.

(3) Let K be the splitting field of $X^n - 1$ over $\mathbb{F}_p$, where n is a positive integer. Pick the correct statement(s) from below.
 (A) K has p^n elements.
 (B) If $p \nmid n$, then the group of field automorphisms of K is isomorphic to the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$.
 (C) K is a separable extension of $\mathbb{F}_p$.
 (D) There exists $m > n$ such that K is the splitting field of $X^m - 1$ over $\mathbb{F}_p$.

(4) Let k be a finite field of characteristic $p > 2$ and G the subgroup of $\mathrm{GL}_2(k)$ consisting of all matrices whose first column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Pick the correct statement(s) from below.
 (A) G is a normal subgroup of $\mathrm{GL}_2(k)$.
 (B) G is a p-group.
 (C) $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in k \right\}$ is a normal subgroup of G.
 (D) G is abelian.

(5) Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \mapsto (-x - y^2, y + x^2)$. Pick the correct statement(s) from below.
 (A) There exist infinitely many $(a, b) \in \mathbb{R}^2$ such that there is an open neighbourhood U of (a, b) such that $f|_U$ is a homeomorphism from U to $f(U)$.
 (B) The derivative Df maps some non-zero tangent vector to $\mathbb{R}^2$ at $(\frac{1}{2}, \frac{1}{2})$, to the zero tangent vector at $(-\frac{3}{4}, \frac{3}{4})$.
 (C) There exist infinitely many $(a, b) \in \mathbb{R}^2$ such that for every open neighbourhood U of (a, b), $f|_U$ is not a homeomorphism from U to $f(U)$.
 (D) For every differentiable curve γ through $(0, 0)$, $f \circ \gamma$ is differentiable curve.

(6) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. For $x \in \mathbb{R}^+$, denote by $\mathrm{FRAC}(x)$ the fractional part of x, i.e., $x - n$ where n is the largest integer that is less than or equal to x. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\mathrm{FRAC}(x/n)}{n}$. Pick the correct statement(s) from below.
 (A) The above series converges for all $x \in \mathbb{R}^+ - \mathbb{Z}$.
 (B) The above series diverges for some non-negative integer x.
 (C) The above series defines a continuous function in a neighbourhood of $\frac{1}{2}$.
 (D) The above series defines a continuous function in a neighbourhood of 1.

(7) Let $f_n(x) = \frac{1}{1+x^n}$. Pick the correct statement(s) from below.
 (A) f_n converges uniformly on $[0, 1/2]$.
 (B) f_n converges uniformly on $[0, 1)$.
 (C) f_n converges uniformly on $[0, 2]$.
 (D) f_n converges pointwise on $[0, \infty)$.

(8) Pick the correct statement(s) from below.
 (A) If $f(z)$ is a function defined on $\mathbb{C}$ that satisfies the Cauchy-Riemann equations at $z = 0$, then $f(z)$ is complex-differentiable at $z = 0$.
 (B) The function $\frac{(\sin z - z)\bar{z}^3}{|z|^6}$ is holomorphic on $\{z \in C : 0 < |z| < 1\}$ and has a removable singularity at $z = 0$.
 (C) There exists a holomorphic function on $\{z \in \mathbb{C} : |z| > 3\}$ whose derivative is $\dfrac{z}{(z-2)^2}$.

(D) There exists a holomorphic function on the upper half plane $\{z \in \mathbb{C} : \Im z > 0\}$ whose derivative is $\dfrac{z}{(z-2)^2(z^2+4)}$.

(9) Pick the correct statement(s) from below.

(A) There exists a maximal ideal M of $\mathbb{Z}[x]$ such that $M \cap \mathbb{Z} = (0)$.

(B) If M is a maximal ideal of $\mathbb{Z}[x]$, then $\mathbb{Z}[x]/M$ is finite.

(C) If I is an ideal of $\mathbb{Z}[x]$ such that $\mathbb{Z}[x]/I$ is finite, then I is maximal.

(D) The ideal $(7, x^2 - 14x - 2)$ in $\mathbb{Z}[x]$ is maximal.

(10) Let $M_n(\mathbb{R})$ be the space of $n \times n$ real matrices. View $M_n(\mathbb{R})$ as a metric space with

$$d([a_{i,j}], [b_{i,j}]) := \max_{i,j}|a_{i,j} - b_{i,j}|.$$

Let $U \subset M_n(\mathbb{R})$ be the subset of matrices $M \in M_n(\mathbb{R})$ such that $(M - I_n)^n = 0$.

(A) U is closed.

(B) U is open.

(C) U is compact.

(D) U is neither closed or open.

Part B

(11) Let G be an abelian group and let H be a nontrivial subgroup of G, that is, H is a subgroup containing at least two elements. Show that the following two statements are equivalent.

(A) For every nontrivial subgroup K of G, the subgroup $K \cap H$ is also nontrivial.

(B) H contains every nontrivial minimal subgroup of G and every element of the quotient group G/H has finite order.

(12) Consider the ring $\mathcal{C}(\mathbb{R})$ of continuous real-valued functions on $\mathbb{R}$, with pointwise addition and multiplication. For $A \subset \mathbb{R}$, the *ideal* of A is $I(A) = \{f \in \mathcal{C}(\mathbb{R}) \mid f(a) = 0 \text{ for all } a \in A\}$. For a subset I of $\mathcal{C}(\mathbb{R})$, the *zero-set* of I is $Z(I) = \{a \in \mathbb{R} \mid f(a) = 0 \text{ for all } f \in I\}$. Prove the following:

(A) (3 marks) $Z(I \cap J) = Z(IJ)$ for ideals I and J of $\mathcal{C}(\mathbb{R})$.

(B) (2 marks) For each $a \in \mathbb{R}$, $I(a)$ is a maximal ideal.

(C) (3 marks) The set $\{f \in \mathcal{C}(\mathbb{R}) \mid f \text{ has compact support}\}$ is a proper ideal, and its zero set is empty.

(D) (2 marks) True/False: For each prime ideal $\mathfrak{p}$ of $\mathcal{C}(\mathbb{R})$, $Z(\mathfrak{p})$ is a singleton set. (Justify your answer.)

(13) Let f, g, h be functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$h(f(x) + g(y)) = xy \qquad (*)$$

for all $x, y \in \mathbb{R}$. Show the following:

(A) (2 marks) h is surjective.

(B) (3 marks) If f is continuous then f is strictly monotone.

(C) (5 marks) There do not exist continuous functions f, g, h satisfying $(*)$.

(14) Let $f : [0,1] \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions. Assume that g is periodic with period 1. Show that

$$\lim_{n \mapsto \infty} \int_0^1 f(x)g(nx)dx = \left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right).$$

(15) Prove or disprove each of the statements below.

(A) (4 marks) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function that takes both positive and negative values. Then f has infinitely many zeros.

(B) (6 marks) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be a continuous function. Then f is not open.

(16) Prove or disprove the following statements:

(A) (5 marks) Suppose that $f(z)$ is a complex analytic function in the punctured unit disk $0 < |z| < 1$ such that $\lim\limits_{n\longrightarrow\infty} f(\frac{1}{n}) = 0$ and $\lim\limits_{n\longrightarrow\infty} f(\frac{2}{2n-1}) = 1$, then there exists a positive integer $N > 0$ such that $\lim\limits_{z\longrightarrow 0} |z^{-N} f(z)| = \infty$.

(B) (5 marks) There exists a non-zero entire function f such that $f(e^{2\pi i e n!}) = 0$ for all $n \geq 2025$.

(17*) Let $X \subseteq \mathbb{R}^n$ and $p \in X$. By a *tangent vector* of X at p, we mean $\gamma'(0)$, where $\gamma : (-\epsilon, \epsilon) \longrightarrow X$ is a differentiable function with $\gamma(0) = p$. ($\epsilon \in \mathbb{R}$, $\epsilon > 0$.) The *tangent space* of X at p is the $\mathbb{R}$-vector space of all the tangent vectors at p. Think of $\mathrm{GL}_n(\mathbb{C})$ as a subspace of $\mathbb{R}^{2n^2}$, with the euclidean topology.

Let $G := \{A \in \mathrm{GL}_2(\mathbb{C}) \mid A^*A = AA^* = I_2, \det A = 1\}$.

(A) (2 marks) Show that every tangent vector of $\mathrm{GL}_n(\mathbb{C})$ at I_n is of the form $\gamma_A'(0)$ where A is a $n \times n$ complex matrix and $\gamma_A : \mathbb{R} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is the function $t \mapsto e^{tA}$.

(B) (3 marks) Show that the tangent space of G at I_2 is $V := \left\{ \begin{bmatrix} ia & z \\ -\bar{z} & -ia \end{bmatrix} \mid a \in \mathbb{R}, z \in \mathbb{C} \right\}$.

(C) (5 marks) Consider the homeomorphism $\Phi : G \longrightarrow \mathbb{S}^3$ (where $\mathbb{S}^3$ denotes the unit sphere in $\mathbb{R}^4$) given by

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \mapsto (\mathfrak{R}(\alpha), \mathfrak{I}(\alpha), \mathfrak{R}(\beta), \mathfrak{I}(\beta))$$

Define a 'multiplication' on V by $[A, B] = \frac{AB-BA}{2}$. Determine the multiplication on the tangent space at $\Phi(I_2)$ induced by the derivative $D\Phi$. (Hint: The map $(A, B) \longrightarrow [A, B]$ is $\mathbb{R}$-bilinear.)

(18*) Let $\mathbb{F}_q$ be the finite field with q elements and $P \in \mathbb{F}_q[x]$ be a monic irreducible polynomial of even degree $2d$. Then show that P, when considered as a polynomial in $\mathbb{F}_{q^2}[x]$, decomposes into a product $P = Q_1Q_2$ of irreducible polynomials Q_i in $\mathbb{F}_{q^2}[x]$ with $\deg(Q_i) = d$.

(19*) Show that the power series $\sum_{n=1}^{\infty} z^{n!}$ represents an analytic function $f(z)$ in the open unit disk Δ centred at 0. Show that $f(z)$ cannot be extended to a continuous function on any connected open set U such that U is strictly larger than Δ.

(20*) It is known that there exist surjective continuous maps $I \longrightarrow I^2$ where $I = [0, 1]$ is the unit interval.

(A) (4 marks) Using the above result or otherwise, show that there exists a surjective continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$.

(B) (6 marks) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be a surjective continuous map. Let $\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3$. Show that $\mathbb{R}^3\ \Gamma$ is path connected.