CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
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Part A

(1) A, B, C.
(2) A, B, D.
(3) C, D.
(4) C.
(5) A, B, C, D.
(6) A, C.
(7) A, D.
(8) B.
(9) B.
(10) A.

Part B

(11) (A) $\Rightarrow$ (B) : Let K be a nontrivial minimal subgroup of G. Then $K \cap H$ is nontrivial and by minimality $K \cap H = K$. This gives the first statement of (B). Let $0 \neq x \in G$. Then $\langle x \rangle \cap H \neq 0$, so that $nx \in H$ for a positive integer n. Here $\langle x \rangle$ denotes the subgroup generated by x in G. This gives the second statement of (B). (B) $\Rightarrow$ (A) : Let K be a nontrivial subgroup of G. To prove that $K \cap H \neq 0$, it is enough to show that $\langle a \rangle \cap H \neq 0$ for some nonzero element $a \in K$. So let us assume that $K = \langle a \rangle$ is cyclic. If K is finite then it will contain a minimal nontrivial subgroup which is contained in H by the first statement of (B). So $K \cap H \neq 0$. Suppose that K is infinite. By hypothesis, $na \in H$ for a positive integer n. Since a has infinite order, we conclude that $0 \neq na \in H \cap K$.

(12) (A) Since $IJ \subseteq I \cap J$, $Z(I \cap J) \subseteq Z(IJ)$. To prove the containment in the other direction, let $a \in Z(IJ)$ and $f \in I \cap J$. We want to show that $f(a) = 0$. Note that $f^2 \in IJ$. Hence $f(a)^2 = 0$. Therefore $f(a) = 0$.

(B) $I(a)$ is the kernel of the surjective ring homomorphism $\mathcal{C}(\mathbb{R}) \longrightarrow \mathbb{R}$, $f \mapsto f(a)$.

(C) If f and g have compact support, so does $f + g$. If f has compact support and g is any continuous function, gf has compact support. Hence the given set is an ideal. If it were not a proper ideal, it would contain the unit, i.e., the constant function 1, which is not compactly supported. For every $a \in \mathbb{R}$, there exists a compactly supported continuous function f such that $f(a) \neq 0$.

(D) False. Choose a maximal ideal $\mathfrak{p}$ containing the ideal I given in (C). Then $Z(\mathfrak{p}) \subseteq Z(I) = \emptyset$.

(13) The relation $h(f(x)+g(y)) = xy$, for all points $x, y \in \mathbb{R}$ imply that h is surjective. If $f(x) = f(y)$ then $x = h(f(x)+g(1)) = h(f(y)+g(1)) = y$, hence f is injective. Also since f is continuous, f is strictly monotone. If f is bounded above then $\lim_{x \mapsto \infty} f(x) = \alpha$ exists, but

$$h(\alpha + g(1)) = \lim_{x \mapsto \infty} h(f(x) + g(1)) = \lim_{x \mapsto \infty} x = \infty.$$

So f is unbounded above, and similarly unbounded below. Thanks to continuity f maps onto $\mathbb{R}$. So f is a bijection. So $f(x)+g(0)$ takes all values in $\mathbb{R}$. But $h(f(x)+g(0))=0$ for all x, which is a contradiction.

(14) $\int_0^1 f(x)g(nx)dx = \frac{1}{n}\int_0^n f\left(\frac{y}{n}\right)g(y)dy = \frac{1}{n}\sum_{k=1}^n \int_k^{k+1} f\left(\frac{y}{n}\right)g(y)dy = \frac{1}{n}\sum_{k=1}^n \int_0^1 f\left(\frac{k+z}{n}\right)g(z)dz,$ by periodicity of g. By replacing g by $g+c$ for some constant c, we can assume g is non-negative. Using continuity and mean value theorem for integrals, we can write the above sum as $\frac{1}{n}\sum_{k=1}^n f\left(\frac{k+z_k}{n}\right)\int_0^1 g(z)dz$ for $z_k \in (0,1)$. Now $\frac{1}{n}\sum_{k=1}^n f\left(\frac{k+z_k}{n}\right)$ converges to the Riemann integral $\int_0^1 f(x)dx$.

(15) (A) True. Let $a\in\mathbb{R}^2$ be such that $f(a)>0$ and $b\in\mathbb{R}^2$ be such that $f(b)<0$. Let ℓ be the line perpendicular to the segment joining a and b passing through the midpoint of the segment. For each $p\in\ell$, there is a straight-line path joining a with p and then p with b. On each such path there is a point q such that $f(q)=0$.

(B) True. Let $U=(a,b)$. Suppose that $f(U)$ is open. Note that $f(U)$ is bounded. Then $\partial f(U)$ is uncountable. On the other hand, since ∂U has exactly two elements, $\partial f(U)$ is finite.

(16) (A) False. The point $z=0$ is an essential singularity. To see this, suppose that for some $n\geq 1, g(z)=z^{-n}f(z)$ is such that $\lim_{z\to 0} g(z)=\infty$, then $z=0$ is a pole of $g(z)$. Let k be the order of the pole of $g(z)$ at $z=0$ and write $g(z)=z^{-k}h(z)$ where $h(z)$ is analytic in $|z|<r$ for some $r\leq 1$ with $h(z)\neq 0$. Then $f(z)=z^{n-k}h(z)$. If $n\geq k$, then $f(z)$ is analytic at $z=0$ and hence continuous at $z=0$, contrary to our assumption on $f(z)$. If $n<k$, then $n-k<0$. As $h(0)\neq 0$, we have $\lim_{z\to 0} f(z)=\infty$, again a contradiction.

(B) False. Write a_n for the fractional part of $en!$. Note that $a_n \longrightarrow 0$. Hence $f(e^{2\pi i a_n})=0$.

(17*) (A) Write $\gamma(t)=I_n+tA+O(t^2)$ where $A=\gamma'(0)\in M_n$. Now consider γ_A.

(B) Every tangent vector of G is also a tangent vector of GL_2, so there exists $A\in M_2$ such that $\gamma_A'(0)=A$ is the tangent vector. $I_2=(I_2+tA+O(t^2))^*(I_2+tA+O(t^2))=I_2+t(A^*+A)+O(t^2)$, so $A^*=-A$. Since $\det\gamma_A(t)=1$ for all t, $\operatorname{Tr} A=0$.

(C) $\Phi(I_2)=(1,0,0,0)$ and the tangent space there is $\mathbb{R}^3$. Let $A=\begin{bmatrix} ia & z\\ -\bar{z} & -ia\end{bmatrix}\in V$. Then

$$\Phi(\gamma_A(t)) = \Phi\left(\begin{bmatrix} 1+iat+O(t^2) & zt+O(t^2)\\ -\bar{z}t+O(t^2) & 1+iat+O(t^2)\end{bmatrix}\right)$$
$$= (1+O(t^2), at+O(t^2), \mathfrak{R}(z)t+O(t^2), \mathfrak{I}(z)t+O(t^2)).$$

Therefore $(D\Phi)_{I_2}(A)=(a,\mathfrak{R}(z),\mathfrak{I}(z))$. Note that the map $(A,B)\longrightarrow[A,B]$ is $\mathbb{R}$-bilinear. An $\mathbb{R}$-basis for V is

$$\left\{\begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix}, \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix} 0 & i\\ i & 0\end{bmatrix}\right\}.$$

Call these A_1,A_2,A_3 respectively. $(D\Phi)_{I_2}(A_i)$ is the ith standard basis vector e_i of $\mathbb{R}^3$. Moreover, $[A_1,A_2]=A_3, [A_2,A_3]=A_1, [A_3,A_1]=A_2$. Hence the induced multiplication on $\mathbb{R}^2$ is the vector cross product.

(18*) Since P is irreducible, $\mathbb{F}_q(\alpha)=\mathbb{F}_{q^{2d}}$ for each root α of P. Thus the splitting field of P over $\mathbb{F}_q$ is $\mathbb{F}_{q^{2d}}$. One has that $\mathrm{Gal}(\mathbb{F}_{q^{2d}}/\mathbb{F}_q)\simeq\mathbb{Z}/(2d)$, generated by the

Frobenius element $\sigma : x \mapsto x^q$. Since the Galois group is cyclic, all roots of P are of the form $\sigma^i(\alpha)$, where α is a chosen root. Then we set

$$Q_1 := \prod_{i \text{even}} (x - \sigma^i(\alpha))$$

and

$$Q_2 := \prod_{i \text{odd}} (x - \sigma^i(\alpha)),$$

we have that $P = Q_1 Q_2$. Each of the polynomials Q_i have coefficients which are symmetric polynomials in their roots, and it is easy to see that these coefficients are fixed by σ^2. The fixed field of $\langle \sigma^2 \rangle$ is $\mathbb{F}_{q^2}$ and thus Q_i has coefficients in $\mathbb{F}_{q^2}$. Since all roots of Q_i lie in a single Galois orbit for $\langle \sigma^2 \rangle$, Q_i is irreducible in $\mathbb{F}_{q^2}[x]$.

(19*) If $0 < \alpha < 1$ is rational $\omega := e^{2\pi i \alpha}$ is a root of unity. So $(r\omega)^{m!} = r^{m!}$ for all $m \geq n$ for $n := o(\omega)$. Summing from the the first N th terms we obtain $s = \sum_{m \geq n} r^{m!}$ and it suffices to show this sum diverges to ∞. Consider the sum up to $m+1$ terms of the above series: $s_m = r^{n!} + \dots + r^{m!}$.

Let $N \geq 1$. Choose r so that $r^{N!} \geq 1/2$. This is possible since $t \mapsto t^{N!}$ is continuous at 1 and so $\lim_{r \to 1^-} r^{N!} = 1$. For this choice of r, we have $s_N \geq (N - n + 1)/2$. As $n = o(\omega)$ is fixed, we see that $s_N \to \infty$ as $N \to \infty$.

(20*) (A) Cover $\mathbb{R}^2$ by unit squares $R_{m,n}$ with sides parallel to the coordinate axes and centres at $(m + 1/2, n + 1/2)$ with $(m, n) \in \mathbb{Z}^2$. It is easily seen that the rectangles $\{R_{m,n} \mid (m, n) \in \mathbb{Z}^2\}$ can be arranged in a sequence $S_0 = R_{0,0}, S_1, S_2, \dots, S_n, \dots$ so that (i) $S_k \neq S_l$ if $k \neq l$, (ii) $S_i \cap S_{i+1}$ is a common edge of both S_i, S_{i+1}.

Suppose that $f_0 : I \to I^2$ is a surjective continuous map. We may arrange so that $f_0(0) = (0, 0) = f(1)$. (Otherwise let $\alpha : I \to I^2$ describe the straight-line segment from $f(0)$ to 0 and $\beta : I \to I^2$, the straight-line segment joining $f(1)$ to $(0, 0)$ and consider the concatenation $f := \alpha \cdot f_0 \cdot \beta : I \to I^2$, which describes α, f_0, β with thrice the speed in succession. Then f maps $0, 1 \in I$ to $(0, 0)$.)

Consider $\phi : \mathbb{R} \to \mathbb{R}^2$ is defined as follows: $\phi(t) = (0, 0)$ if $t \leq 0, \phi(t) = f_0(t), 0 \leq t \leq 1$, and , inductively, having defined ϕ on $[j-1, j]$ such that $\phi([j-1, j]) = S_{j-1} = R_{m,n}$, with $\phi(j-1) = (m, n)$, we proceed to define ϕ on $[j, j+1]$ as follows. By our hypothesis, $S_j = R_{k,l}$ where $(k, l) \in \{(m, n+1), (m, n-1), (m+1, n), (m-1, n)\}$. Let $\sigma : I \to R_{m,n} \cup R_{k,l}$ be the straight line segment that joins (m, n) to (k, l). Then define

$$\phi(j + t) = \begin{cases} \sigma(2t), & \text{if } 0 \leq t \leq 1/2 \\ (k, l) + f_0(2t - 1), & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Then $\phi : \mathbb{R} \to \mathbb{R}^2$ is a continuous surjection. (Continuity follows from pasting lemma and surjectivity follows from the fact that $\bigcup R_{m,n} = \mathbb{R}^2$.)

(B) The complement of the graph of ϕ is path connected. Suppose that both $p_0 = (x_0, y_0, z_0), p_1 = (x_1, y_1, z_1)$ are not in Γ. If $x_0 = x_1$, then p_0, p_1 belong the plane P with equation $x = x_0$ and $\Gamma \cap P$ is a singleton. So $P \backslash \Gamma$ is a punctured plane, which is path connected.

Suppose that $x_0 < x_1$. Then $\Gamma \cap \{(t, y, x) \mid x_0 \leq t \leq x_1\}$ is compact. So we can choose a $c > 0$ large such that Γ is contained in the half space $\{(x, y, z) \mid z < c\}$. Now the straight-line segment joining $q_0 = (x_0, 0, c)$ and

$q_1 = (x_1, 0, c)$ does not meet Γ. Applying first case (twice) we get a path from p_0 to q_0 to q_1 to p_1 not meeting Γ.