CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
19th May 2024

Part A

(1) Let $f(z) = z^7 - 4z^3 - 11$. Pick the correct statement(s) from below.
   (A) $f(z)$ has at least 1 zero in the open set $\{|z| > 2\}$.
   (B) $f(z)$ has at least 5 zeroes in the annular region $\{1 < |z| < 2\}$.
   (C) $f(z)$ has exactly 6 zeroes in the annular region $\{1 < |z| < 2\}$.
   (D) $f(z)$ has exactly 1 zero in the closed disc $\{|z| \leq 1\}$.

(2) A *region* in $\mathbb{C}$ is a non-empty open connected set. Select all the statement(s) that are true.
   (A) Let f be a function on a region Ω such that the integral of f along the boundary of any closed triangle in Ω is zero. Then f is analytic on Ω.
   (B) There exist a region Ω containing the real interval $(0, 1)$ and a non-zero analytic function $f : \Omega \to \mathbb{C}$ such that $f\left(\frac{1}{n}\right) = 0$ for all positive integers n.
   (C) Let f be an analytic function on $\mathbb{C} \setminus \{0\}$ with an essential singularity at $z = 0$. Then $\lim_{z \to 0} |f(z)| = \infty$.
   (D) Every bounded analytic function on $\mathbb{C} \setminus \{0\}$ is constant.

(3) Let u and v be real-valued functions on $\mathbb{R}^2$ defined as follows:

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

$$v(x, y) = \begin{cases} \frac{y^3 - 3yx^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function $f(x, y) = (u(x, y), v(x, y))$. Pick the correct statement(s) from below.
   (A) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist at $(0, 0)$.
   (B) $\frac{\partial u}{\partial x}$ is continuous at $(0, 0)$.
   (C) For every fixed $(a, b) \neq (0, 0) \in \mathbb{R}^2$, the function $t \mapsto f(ta, tb)$ is a differentiable function (of t).
   (D) f is differentiable at $(0, 0)$.

(4) Let G (respectively, H) be a Sylow 2-subgroup (respectively, Sylow 7-subgroup) of the symmetric group S_{17}. Pick the correct statement(s) from below.
   (A) The order of G is 2^{15}.
   (B) H is abelian.
   (C) G has a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.
   (D) If $\sigma \in S_{17}$ has order 4, then σ is a 4-cycle.

(5) Let $p \geq 3$ be a prime number and V be an n-dimensional vector space over $\mathbb{F}_p$. Let $T : V \to V$ be a linear transformation. Select all the true statement(s) from below.
   (A) T has an eigenvalue in $\mathbb{F}_p$.
   (B) If $T^{p-1} = I$, then the minimal polynomial of T has distinct roots in $\mathbb{F}_p$.
   (C) If $T \neq I$ and $T^{p-1} = I$, then the characteristic polynomial of T has distinct roots in $\mathbb{F}_p$.
   (D) If $T^{p-1} = I$, then T is diagonalizable over $\mathbb{F}_p$.

(6) Let X be a subset of $\mathbb{R}^3$. We say that X has property S if it contains at least two elements and every Cauchy sequence in X has a limit point in X. Pick the correct statement(s) from below.
   (A) If X has property S then it must be compact.
   (B) If X has property S then it must be closed.
   (C) Suppose that X has property S and it further satisfies the following condition: if $(a_1, b_1, c_1), (a_2, b_2, c_2) \in X$, then $(a_1 + a_2, b_1 + b_2, c_1 +, c_2) \in X$. Then X is dense in $\mathbb{R}^3$.
   (D) Suppose that X has property S and it further satisfies the following condition: if $(a, b, c) \in X$, then $(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \in X$. Then X is dense in $\mathbb{R}^3$.

(7) Let f be a continuous real-valued function on $[0, 1]$ such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 0.$$

Pick the correct statement(s) from below.

(A) f must have a zero in $[0, 1]$.
(B) f has at least two zeros, counted with multiplicity, in $[0, 1]$.
(C) If $f \not\equiv 0$, then f has exactly two zeros in $[0, 1]$.
(D) $f \equiv 0$.

(8) Which of the following statement(s) are true?
(A) Every prime ideal of a finite commutative ring with unity is maximal.
(B) A commutative ring with unity whose set of all ideals is countably infinite is necessarily a countable ring.
(C) Let R be a unique factorisation domain and K be its field of fractions. There exists an irreducible element $\alpha \in R$ and an element $\beta \in K$ such that $\beta^2 = \alpha$.
(D) Every subring R (with unity) of $\mathbb{Q}$ with $\mathbb{Z} \subsetneq R \subsetneq \mathbb{Q}$ has infinitely many prime ideals.

(9) Which of the following statement(s) are true?
(A) If F_1, F_2 are finite field extensions of $\mathbb{Q}$ such that $[F_1 : \mathbb{Q}] = [F_2 : \mathbb{Q}]$, then F_1, F_2 are isomorphic as fields.
(B) If F_1, F_2 are finite field extensions of $\mathbb{R}$ such that $[F_1 : \mathbb{R}] = [F_2 : \mathbb{R}]$, then F_1, F_2 are isomorphic as fields.
(C) Let $\mathbb{F}$ be a finite field. If F_1, F_2 be finite field extensions of $\mathbb{F}$ such that $[F_1 : \mathbb{F}] = [F_2 : \mathbb{F}]$, then F_1, F_2 are isomorphic as fields.
(D) Let $\omega \in \mathbb{C}$ be a primitive cube root of unity and let $\sqrt[3]{2} \in \mathbb{R}$ be a cube root of 2. Let $K = \mathbb{Q}(\omega, \sqrt[3]{2})$. If F_1, F_2 are subfields of K such that $[K : F_1] = [K : F_2] = 2$, then $F_1 = F_2$.

(10) Let $X = \{\alpha \in \mathbb{C} \mid \alpha \text{ satisfies a monic polynomial over } \mathbb{Q}\}$. (I.e., X is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.) Endow X with the subspace topology of the euclidean metric topology from $\mathbb{C}$. Pick the correct statement(s) from below.
(A) X is closed.
(B) X is complete.
(C) X is unbounded.
(D) X is connected.

Part B

(11) Let $n \geq 3$ be an integer. Write D_{2n} for the dihedral group with $2n$ elements. Show that the automorphism group of D_{2n} has at most $n\varphi(n)$ elements. (Here $\varphi(n)$ is the number of positive integers that are relatively prime to n.)

(12) Let $\mathrm{SL}(2, \mathbb{R})$ be the group of 2×2 matrices with real entries and determinant 1, endowed with the subspace topology of $\mathbb{R}^4$. Consider the continuous map $f : \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathbb{C}$ given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{ai + b}{ci + d}.$$

(A) (4 marks) Show that f maps $\mathrm{SL}(2, \mathbb{R})$ onto the upper half plane $H = \{z : \mathrm{Im}(z) > 0\}$.
(B) (6 marks) Assume the following two facts:
(i) For all $M, N \in \mathrm{SL}(2, \mathbb{R})$, $f(M) = f(N)$ if and only if $M^{-1}N$ is an orthogonal matrix.
(ii) The map f is an open map.
Now show that for every compact $K \subseteq H$, $f^{-1}(K)$ is compact.

(13) Let G be a finite group of odd order and $1 < d < |G|$ be a divisor of $|G|$. Assume that G has exactly three subgroups H_1, H_2 and H_3 of order d. Suppose that H_1 is not normal in G. For each $i = 1, 2, 3$, let N_i denote the normalizer of H_i. Let $S := \{H_1, H_2, H_3\}$.
(A) (4 marks) For each $g \in G$ let s_g denote the cardinality of the set $\{H \in S \mid gHg^{-1} = H\}$. Show that $\sum_{g \in G} s_g = |N_1| + |N_2| + |N_3|$.
(B) (6 marks) Show that $G \neq N_1 \cup N_2 \cup N_3$.

(14) (A) (5 marks) Let X be a non-empty finite set and let R be the ring of $\mathbb{Z}$-valued functions on X, with pointwise addition and multiplication. Let S be an additive subgroup of R such that the multiplicative identity $1_R \notin S$ and such that for all $s, s' \in S$, $ss' \in S$. Show that there exists $x \in X$ and a prime number p such that $f(x)$ is divisible by p for all $f \in S$ (Hint: Consider the sets $\{f(x) : f \in S\}$ for all $x \in X$.)
(B) (5 marks) Let K be a subfield of $\mathbb{C}$ with $[K : \mathbb{Q}] = 2$. Let $P \in \mathbb{Q}[x]$ be irreducible over $\mathbb{Q}$. Show that P is either irreducible in $K[x]$ or splits as product of two irreducible polynomials in $K[x]$.

(15) Let $A(X), B(X)$ be non-zero polynomials in $\mathbb{C}[X]$ such that $0 \leq \deg A \leq \deg B - 2$ and $A(X)$ and $B(X)$ do not share any roots. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the roots of $B(X)$. Suppose that each of them is a simple root.

Show that

$$\sum_{j=1}^{k} \frac{A(\alpha_j)}{B'(\alpha_j)} = 0.$$

(16) (A) (3 marks) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(r) = f(r + \frac{1}{n})$ for each $r \in \mathbb{Q}$ and each positive integer n. Prove or disprove the following statement: f is a constant function.

(B) (7 marks) Let $a_n, n \geq 1$ be a sequence of non-negative real numbers such that $a_{m+n} \leq a_m + a_n$ for all m, n. Show that

$$\lim_{n \longrightarrow \infty} \frac{a_n}{n} = \inf \left\{ \frac{a_n}{n} \mid n \geq 1 \right\}.$$

(17*) Let p be a prime number. Let $n \geq 2$ be an integer. Let V be an n-dimensional $\mathbb{F}_p$-vector space. Determine, with a proof, the number of two-dimensional $\mathbb{F}_p$-subspaces of V.

(18*) Let R be the ring of all the real-valued functions on $\mathbb{N} \times \mathbb{N}$. Show that R contains a subring isomorphic to the polynomial ring $\mathbb{R}[X, Y]$.

(19*) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real entries, identified with the euclidean space $\mathbb{R}^{n^2}$. Let X be a compact subset of $M_n(\mathbb{R})$, and $S \subset \mathbb{C}$ be the set of all eigenvalues of the matrices in X. Show that S is a compact subset of $\mathbb{C}$.

(20*) Fix $0 < \lambda < 1$. Choose $\epsilon > 0$ such that $\epsilon + \frac{\lambda^2}{4} \leq \frac{\lambda}{2}$. Consider the metric space

$$X := \{\psi \in \mathcal{C}^1([-\epsilon, \epsilon]) : |y + \psi(y)| \leq \frac{\lambda}{2} \text{ for all } y \in [-\epsilon, \epsilon]\},$$

with the induced supremum metric from $\mathcal{C}^1([-\epsilon, \epsilon])$, which we denote by d. (Recall: $\mathcal{C}^1([-\epsilon, \epsilon])$ is the set of real-valued differentiable functions on $[-\epsilon, \epsilon]$ whose derivative is continuous.)

(A) (1 mark) Show that there is a function $A : X \longrightarrow X$ given by

$$(A\psi)(y) = -(y + \psi(y))^2.$$

(B) (2 marks) Show that if $\psi \in X$ is such that $A\psi = \psi$, then the function $x = y + \psi(y)$ is an inverse to the function $y = x + x^2$, locally near the origin.

In the next few steps, we show that such a ψ exists.

(C) (2 marks) Show that $d(A\psi_1, A\psi_2) \leq \lambda d(\psi_1, \psi_2)$.

(D) (4 marks) Let $\phi \in X$. Show that the sequence $A^n\phi, n \geq 1$ is a Cauchy sequence, and it has a limit. By A^n, we mean the n-fold composition $A \circ A \circ \cdots \circ A$. (You may use the fact that X is complete with respect to d.)

(E) (1 mark) Show that there exists $\psi \in X$ such that $A\psi = \psi$.