CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
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Part A

(1) B
(2) B, D.
(3) A, C.
(4) A, B, C.
(5) B, D.
(6) B.
(7) A, B.
(8) A.
(9) B, C.
(10) C.

Part B

(11) Write $D_{2n} = \langle \sigma, \tau \mid \sigma^n = e, \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$.

Let $\Phi \in \mathrm{Aut}(D_{2n})$. Note that Φ is determined by $\Phi(\sigma)$ and $\Phi(\tau)$. Since $\Phi(\sigma)^n = \Phi(\sigma^n) = e$, we find that $\Phi(\sigma)$ has order dividing n. Since Φ is injective, $\Phi(\sigma^d) \neq e$ for $d < n$. Hence, the order of $\Phi(\sigma)$ is exactly n. Every element of D_{2n} is either of the form $\tau\sigma^j$ or σ^i. Note that $\tau\sigma^j$ has order 2. Since $n > 2$, we find that $\Phi(\sigma) \neq \tau\sigma^j$. Therefore, there is an index i such that $\Phi(\sigma) = \sigma^i$. The order of σ^i is $n/\gcd(n,i)$, and therefore, i is coprime to n. Thus, there are $\varphi(n)$ choices for $\Phi(\sigma)$. The elements in D_{2n} that have order 2 are all of the form $\tau\sigma^j$, where $j \in \{0, 1, \dots, n-1\}$, or if n is even, $\sigma^{n/2}$. Note that since $\Phi(\sigma) \in \langle \sigma \rangle$, it follows that Φ would fail to be surjective if $\Phi(\tau) = \sigma^{n/2}$. Thus, $\Phi(\tau)$ is an element of the form $\tau\sigma^j$, with $j \in \{0, 1, \dots, n-1\}$. Hence $|\mathrm{Aut}(D_{2n})| \leq n\varphi(n)$.

(12) (A)
$$\frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{ac+bd+(ad-bc)i}{c^2+d^2} = \frac{ac+bd+i}{c^2+d^2},$$
so it is in H. Given $x + yi \in H$, choose a, b, d such that $x = b/d$, $y = 1/d^2$ and $ad = 1$. Now
$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = x + yi.$$

(B) Let K be a compact subset of H. For each $M \in \mathrm{SL}(2,\mathbb{R})$, choose a compact neighbourhood $V(M)$ of M. Since f is open and the sets $f(V(M)), M \in \mathrm{SL}(2,\mathbb{R})$ cover K, a finitely many subcollection, say $f(V(M_1)), \dots, f(V(M_n))$ will cover K. Let $L = \bigcup_{i=1}^n V(M_i)$; it is a compact subset of $\mathrm{SL}(2,\mathbb{R})$. Now $f^{-1}(K) \subset L \cdot SO(2,\mathbb{R})$. Now consider the multiplication map $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SL}(2,\mathbb{R})$ which is continuous. Note that $L \cdot SO(2,\mathbb{R})$ is the image of the compact subset $L \times SO(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Hence $L \cdot SO(2,\mathbb{R})$ is compact, so the closed subset $f^{-1}(K)$ is compact.

(13) (A) $\sum_{g \in G} s_g = |\{(g,H) \in G \times S \mid gHg^{-1} = H\}| = \sum_{i=1}^3 |N_i|$.

(B) Consider the action of G on S by conjugation. The orbit of H_1 is equal to $\{H_1, H_2, H_3\}$ by hypothesis. So the action is transitive. Hence the order of each N_i is $|G|/3$. Note that $s_e = 3$ where e is the identity element of G. So by the solution to (A), $s_g = 0$ for some g.

(14) (A) For each $x \in X$, the set $Y_x := \{f(x) : f \in S\}$ is an ideal of $\mathbb{Z}$. We are done if $Y_x \neq \mathbb{Z}$ for some $x \in X$. If not, then $Y_x = \mathbb{Z}$ for all $x \in X$ and hence for every $x \in X$, there exists $f_x \in S$ such that $f_x(x) = 1$. Then
$$\prod_{x \in X} (f_x - 1_R) = 0 \in R.$$
Expand this to write 1_R as a polynomial expression in $\{f_x \mid x \in X\}$. Since every polynomial expression in $\{f_x \mid x \in X\}$ belongs to S and S is an additive subgroup of R, it follows that $1_R \in S$. This is a contradiction. Hence we have $Y_x \subsetneq \mathbb{Z}$ for some $x \in X$.

(B) Assume that P is reducible in $K[x]$. Let g be an irreducible factor of P in $K[x]$. It suffices to show that $\deg g = (\deg P)/2$. Let α a root of g. Now
$$[K(\alpha) : K][K : \mathbb{Q}] = [K(\alpha) : \mathbb{Q}] = [K(\alpha) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}],$$

so $2\deg(g) = \deg(P)[K(\alpha) : \mathbb{Q}(\alpha)]$. Since $\deg g < \deg P$, it follows that $[K(\alpha) : \mathbb{Q}(\alpha)] < 2$ so $[K(\alpha) : \mathbb{Q}(\alpha)] = 1$. Hence $\deg(g) = \deg(P)/2$.

(15) Write $f(z)$ for the meromorphic function $\frac{A(z)}{B(z)}$.

We first note that

$$\frac{A(\alpha_i)}{B'(\alpha_i)} = \operatorname{Res}_{\alpha_i} f(z).$$

Hence we need to show that

$$\sum_{i=1}^{k} \operatorname{Res}_{\alpha_i} f(z) = 0$$

Solution 1: Expand $\frac{A(X)}{B(X)}$ as partial fractions

$$\frac{A(X)}{B(X)} = \sum_{i=1}^{k} \frac{a_i}{X - \alpha_i}$$

We can assume that B is monic, i.e., $B = \prod_{i=1}^{k}(X - \alpha_i)$. Hence

$$A(X) = \sum_{i=1}^{k} a_i \prod_{j \neq i}(X - \alpha_j).$$

from which it follows that the coefficient of X^{k-1} in $A(X) = \sum_{i=1}^{k} a_i$. Since $\deg A(X) < k - 1$, it follows that $\sum_{i=1}^{k} a_i = 0$. On the other hand,

$$a_i = \operatorname{Res}_{\alpha_i} \left(\sum_{j=1}^{k} \frac{a_j}{z - \alpha_j} \right).$$

Hence

$$\sum_{i=1}^{k} \operatorname{Res}_{\alpha_i} f(z) = \sum_{i=1}^{k} a_i = 0.$$

Solution 2:

For each real number $R > \max_i |\alpha_i|$, define

$$I_R := \int_{C_R} f(z) dz,$$

where C_R is the circle of radius R with centre at 0, oriented counter-clockwise. Hence

$$I_R = 2\pi i \sum_{i=1}^{k} \operatorname{Res}_{\alpha_i} f(z).$$

On the other hand, since $\deg A(X) < \deg B - 1$, it follows that

$$|I_R| \leq \int_{C_R} |f(z)| dz \leq \frac{2\pi}{R}$$

Hence

$$\sum_{i=1}^{k} \operatorname{Res}_{\alpha_i} f(z) = 0.$$

(16) (A) Let $r < r'$ be rational numbers and write $r - r' = \frac{m}{n}$ for some positive integers m, n. Then $f(r) = f(r + \frac{1}{n}) = f(r + \frac{2}{n}) = \cdots = f(r')$. Therefore there exists $c \in \mathbb{R}$ such that $f(r) = c$ for all $r \in \mathbb{Q}$. Now let $r \in \mathbb{R}$. Then there exists a sequence $r_k \in \mathbb{Q}, k \geq 1$ converging to r. Since f is continuous, $f(r) = \lim_k f(r_k) = c$. Hence f is a constant function.

(B) Let $\alpha = \inf_n \frac{a_n}{n}$. Then for any $\epsilon > 0$, there exists N such that $a_N < N(\alpha + \epsilon)$. Let $\beta = \max\{a_1, \cdots a_N\}$. Let $n > N$. Write $n = Nq + r$ with $0 \leq r < N$. By the sub-additivity of a_n,

$$a_n \leq q a_N + a_r \leq q a_N + \beta$$

and hence

$$\alpha \leq \frac{a_n}{n} \leq \frac{q a_N}{n} + \frac{\beta}{n} < \frac{qN(\alpha + \epsilon)}{n} + \frac{\beta}{n} \longrightarrow \alpha + \epsilon$$

since $\frac{qN}{n} \longrightarrow 1$ as $n \longrightarrow \infty$, Hence

$$\lim \frac{a_n}{n} = \alpha.$$

(17*) Consider the set $X = \{(v_1, v_2) \mid v_1, v_2$ are linearly independent vectors in $V\}$. Then $|X| = (p^n - 1)(p^n - p)$. For a two-dimensional subspace W of V, define $X_W := \{(v_1, v_2) \in X \mid v_1, v_2$ is a basis of $W\}$. For each two-dimensional subspace W of V, $\mathrm{GL}_2(\mathbb{F}_p)$ acts transitively and without fixed points on X_W. Moreover $X = \cup_W X_W$ where W runs over all the two-dimensional subspaces of V. Hence the orbits of the action on X are in bijective correspondence with the set of two-dimensional $\mathbb{F}_p$-subspaces of V. Hence the number of two-dimensional $\mathbb{F}_p$-subspaces of V is

$$\frac{(p^n - 1)(p^n - p)}{(p^2 - 1)(p^2 - p)}.$$

(18*) Let $s, t : \mathbb{N}^2 \longrightarrow \mathbb{R}$ be the restrictions of the first and second projections $\mathbb{R}^2 \longrightarrow \mathbb{R}$. We now show that the $\mathbb{R}$-subalgebra of R generated by s, t is isomorphic to a polynomial ring in two variables. To do this, it suffices to show that the natural map $\mathbb{R}[X, Y] \longrightarrow R$, $X \mapsto s, Y \mapsto t$ is injective. Let $f(X, Y)$ be in the kernel of this map, i.e, $f(s, t) \equiv 0$. We want to show that $f(X, Y)$ is the zero polynomial. By way of contradiction, assume that it is non-zero. Write $f(X, Y) = \sum_{k=0}^{d} f_k(X) Y^k$ for some suitable d. For each $0 \leq k \leq d$, $f_k(X)$ has only finitely many zeros. Hence there exists $n \in \mathbb{N}$ such that $f(n, Y)$ is a non-zero polynomial. Therefore there exists $m \in \mathbb{N}$ such that $f(n, m) \neq 0$. Thus $f(s, t)$ is non-zero at (n, m), a contradiction. Therefore $f(X, Y)$ is the zero polynomial.

(19*) We show that the set S is closed and bounded. First, we show that S is bounded. Let $\lambda \in S$, there exists a non-zero vector $v \in \mathbb{R}^n$ and a matrix $A = (a_{i,j}) \in X$ such that $Av = \lambda v$. Then, we find that

$$|\lambda||v_i| = |\sum_j a_{i,j} v_j| \leq \max\{|a_{i,j}| : 1 \leq i, j \leq n\} \times \max\{|v_j| \mid 1 \leq j \leq n\}.$$

Taking the maximum value of $|v_i|$, we thus find from the above that

$$|\lambda| \leq \max\{|a_{i,j}| : 1 \leq i, j \leq n\}.$$

Since X is a compact subset of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$, it is bounded. Hence, there exists $D > 0$ such that $\max\{|a_{i,j}| : 1 \leq i, j \leq n\} \leq D$ for all $A \in X$. Thus, we have shown that S is bounded.

In order to show that S is closed, take a sequence $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots$ in S which converges to $\lambda \in \mathbb{C}$. We show that $\lambda \in S$. For each λ_i, there is a non-zero vector v_i and $A_i \in X$ such that $A_i v_i = \lambda_i v_i$. Assume without loss of generality that $|v_i| = 1$ for all i. Since X is compact, there is a subsequence A_{n_i} such that A_{n_i} converges to $A \in X$. Since v_{n_i} all have norm 1, it follows that after passing to a subsequence if necessary, we can assume without loss of generality that v_i converge to a vector v of norm 1. Thus, we find that $Av = \lambda v$. Since $A \in X$, it follows that $\lambda \in S$. This shows that S is closed. Being a closed and bounded subset of $\mathbb{C}$, we find that S is compact.

(20*) (A) $|y + (A\psi)(y)| = |y - (y + \psi(y))^2| \leq |y| + |(y + \psi(y))^2| \leq \epsilon + \frac{\lambda^2}{4} \leq \frac{\lambda}{2}$.

(B) If we compose the two functions, we get the identity map. More precisely, $y = x + x^2 = (y + \psi(y)) + (y + \psi(y))^2 = (y + \psi(y)) - \psi(y) = y$.

(C)

$$\begin{aligned} d(A\psi_1, A\psi_2) &= \sup\{|(y + \psi_1(y))^2 - (y + \psi_2(y))^2| : y \in [-\epsilon, \epsilon]\} \\ &= \sup\{|(2y + \psi_1(y) + \psi_2(y))(\psi_1(y) - \psi_2(y))| : y \in [-\epsilon, \epsilon]\} \\ &\leq \lambda d(\psi_1, \psi_2) \end{aligned}$$

(D) Let n, k be positive integers. Then

$$\begin{aligned} d(A^n\phi, A^{n+k}\phi) &\leq \lambda d(A^{n-1}\phi, A^{n-1+k}\phi) \leq \ldots \leq \lambda^n d(\phi, A^k\phi) \leq \\ &\lambda^n \left(d(\phi, A\phi) + d(A\phi, A^2\phi) + d(A^2\phi, A^3\phi) + \cdots + d(A^{k-1}\phi, A^k\phi)\right) \leq \\ &\lambda^n \left(1 + \lambda + \cdots + \lambda^{k-1}\right) d(\phi, A\phi) \leq \lambda^n (1 + \lambda + \cdots) d(\phi, A\phi) = \frac{\lambda^n}{1 - \lambda} d(\phi, A\phi). \end{aligned}$$

Hence this is a Cauchy sequence. Since X is complete, the sequence has a limit. (Proof that X is complete: It suffices to show that X is closed in $\mathcal{C}^1([-\epsilon, \epsilon])$, since closed subsets of complete spaces are complete. Consider the continuous function

$$F : \mathcal{C}^1([-\epsilon, \epsilon]) \longrightarrow \mathcal{C}^1([-\epsilon, \epsilon]) \qquad \phi \mapsto \mathrm{id} + \phi.$$

Composing this with the sup-norm function gives a continuous map $G : \mathcal{C}^1([-\epsilon, \epsilon]) \longrightarrow \mathbb{R}$. Then $X = G^{-1}([0, \frac{\lambda}{2}])$.)

(E) Let $\phi \in X$. Since A is continuous, we see that

$$A\left(\lim_n A^n\phi\right) = \lim_n A^{n+1}\phi = \lim_n A^n\phi.$$

Hence take $\psi = \lim_n A^n\phi$.