CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
7th May 2023

Part A

(1) Let R be an integral domain containing $\mathbb{C}$ such that it is a finite-dimensional $\mathbb{C}$-vector-space. Pick the correct statement(s) from below.
 (A) For every $a \in R$, the set $\{1, a, a^2, \ldots\}$ is linearly dependent over $\mathbb{C}$.
 (B) R is a field.
 (C) $R = \mathbb{C}$.
 (D) The transcendence degree of R over $\mathbb{C}$ is 1.

(2) Let R be a euclidean domain that is not a field. Let $d : R \setminus \{0\} \longrightarrow \mathbb{N}$ be the euclidean size (degree) function. Write $R^\times$ for the invertible elements of R. Pick the correct statements from below.
 (A) $R = R^\times \cup \{0\}$.
 (B) There exists $a \in R \setminus (R^\times \cup \{0\})$ such that $d(a) = \inf\{d(r) \mid r \in R \setminus (R^\times \cup \{0\})\}$.
 (C) With a defined as above, for all $r \in R$, there exists $u \in R^\times \cup \{0\}$ such that a divides $(r - u)$.
 (D) With a defined as above, the ideal generated by a is a maximal ideal.

(3) Let X be a compact topological space. Let $f : X \longrightarrow \mathbb{R}$ be a function satisfying $f^{-1}([n, \infty))$ is closed for all $n \in \mathbb{N}$. Pick the correct statements from below.
 (A) f is continuous.
 (B) $f(U)$ is open for each open subset U of X.
 (C) $f(U)$ is closed for each closed subset U of X.
 (D) f is bounded above.

(4) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function and $E \subseteq [0, 1]$. Which of the following are true?
 (A) If E is closed, then $f(E)$ is closed.
 (B) If E is open, then $f(E)$ is open.
 (C) If E is a countable union of closed sets, then $f(E)$ is a countable union of closed sets.
 (D) If f injective and E is a countable intersection of open sets, then $f(E)$ is a countable intersection of open sets.

(5) Consider the real matrix
$$A = \begin{pmatrix} \lambda & 2 \\ 3 & 5 \end{pmatrix}.$$
Assume that -1 is an eigenvalue of A. Which of the following are true?
 (A) The other eigenvalue is in $\mathbb{C} \setminus \mathbb{R}$.
 (B) $A + I_2$ is singular.
 (C) $\lambda = 1$.
 (D) Trace of A is 5.

(6) Let a_n, $n \geq 1$, be a sequence of positive real numbers such that $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Then which of the following are true?
 (A) There exists a natural number M such that
$$\sum_{n=1}^{\infty} \frac{1}{(a_n)^M} \in \mathbb{R}.$$
 (B)
$$\sum_{n=1}^{\infty} \frac{1}{(n^2 a_n)} \in \mathbb{R}.$$
 (C)
$$\sum_{n=1}^{\infty} \frac{1}{(n a_n)} \in \mathbb{R}.$$

(D) For all positive real numbers R,

$$\sum_{n=1}^{\infty} \frac{R^n}{(a_n)^n} \in \mathbb{R}.$$

(7) Let A be the ring of all entire functions under point-wise addition and multiplication. Then which of the following are true?

(A) A does not have non-zero nilpotent elements.

(B) In the group of the units of A (under multiplication), every element other than 1 has infinite order.

(C) For every $f \in A$, there is a sequence of polynomials which converges to f uniformly on compact sets.

(D) The ideal generated by z and $\sin z$ is principal.

(8) Which of the following groups are cyclic?

(A) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$

(B) $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$

(C) Every group of order 18.

(D) $(\mathbb{Q}^{\times}, \cdot)$

(9) Let p, q be distinct prime numbers and let ζ_p, ζ_q denote (any) primitive p-th and q-th roots of unity, respectively. Choose all the correct statements.

(A) $\zeta_{13} \notin \mathbb{Q}(\zeta_{31})$.

(B) If p divides $q-1$, then $\zeta_p \in \mathbb{Q}(\zeta_q)$.

(C) If $\zeta_p \in \mathbb{Q}(\zeta_q)$, then $p-1$ divides $q-1$.

(D) If there exists a field homomorphism $\mathbb{Q}(\zeta_p) \longrightarrow \mathbb{Q}(\zeta_q)$, then $p-1$ divides $q-1$.

(10) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be functions. Let $F = (f, g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Assume that F is infinitely differentiable and that $F(0,0) = (0,0)$. Suppose further that the function $fg : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is everywhere non-negative. Then

(A) $f_x(0,0) = 0, f_y(0,0) = 0$.

(B) $g_x(0,0) = 0, g_y(0,0) = 0$.

(C) The image of F is not dense in $\mathbb{R}^2$.

(D) $\det J(0,0) = 0$ where J is the matrix of first partial derivatives (i.e., the jacobian matrix).

Part B

(11) Let $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1, & x = 0 \\ x^{-x}, & x > 0 \end{cases}$$

Determine whether the following statement is true:

$$\int_0^1 f(x) \mathrm{d}x = \sum_{i=0}^{\infty} n^{-n}.$$

(12) (A) (3 marks) Let G be a group such that $|G| = p^a d$ with $a \geq 1$ and $(p, d) = 1$. Let P be a Sylow p-subgroup and let Q be any p-subgroup of G such that Q is not a subgroup of P. Show that PQ is not a subgroup of G.

(B) (7 marks) Let Γ be a group that is the direct product of its Sylow subgroups. Show that every subgroup of Γ also satisfies the same property.

(13) (A) (5 marks) Let $n \geq 2$ be an integer. Let V be the $\mathbb{R}$-vector-space of homogeneous real polynomials in three variables X, Y, Z of degree n. Let $p = (1,0,0)$. Let

$$W = \{f \in V \mid f(p) = \frac{\partial f}{\partial X}(p)\}$$

Determine the dimension of V/W.

(B) (5 marks) A linear transformation $T : \mathbb{R}^9 \longrightarrow \mathbb{R}^9$ is defined on the standard basis $e_1, \ldots, e_9$ by

$$\begin{aligned} Te_i &= e_{i-1}, \quad i = 3, \ldots, 9 \\ Te_2 &= e_3 \\ Te_1 &= e_1 + e_3 + e_8. \end{aligned}$$

Determine the nullity of T.

(14) Let F be a field and R a subring of F that is not a field. Let x be a variable. Let $S = \{a_0 + a_1x + \cdots + a_nx^n \mid n \geq 0 \text{ and } a_0 \in R, a_1, \cdots a_n, \in F\}$.

(A) (2 marks) Show that, with the natural operations of addition and multiplication of polynomials, S is an integral domain.

(B) (4 marks) Let $I = \{f(x) \in S | f(0) = 0\}$. Determine whether I is a prime ideal.

(C) (4 marks) Determine whether S is a PID.

(15) (A) (6 marks) Let $f, g : [0, 1] \mapsto \mathbb{R}$ be monotonically increasing continuous functions. Show that

$$\left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right) \leq \int_0^1 f(x)g(x)dx.$$

(Hint: try double integrals.)

(B) (4 marks) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an infinitely differentiable function such that $f(1) = f(0) = 0$. Also, suppose that for some $n > 0$, the first n derivatives of f vanish at zero. Then prove that for the $(n+1)$th derivative of f, $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$.

(16) (A) (5 marks) Consider the eucliean space $\mathbb{R}^n$ with the usual norm and dot product. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be such that

$$||\mathbf{x} + t\mathbf{y}|| \geq ||\mathbf{x}||, \text{ for all } t \in \mathbb{R}.$$

Show that $\mathbf{x} \cdot \mathbf{y} = 0$.

(B) (5 marks) Consider the vector field $\vec{v} = (v_x, v_y)$ (with components (v_x, v_y)) on $\mathbb{R}^2$:

$$v_x(x, y) = x - y, \; v_y(x, y) = y + x$$

Compute the line integral of $\vec{v}$ along the unit circle (counterclockwise). Is there a function f such that $\vec{v} = \text{grad} f$?

(17*) Denote by V the $\mathbb{Q}$-vector-space $\mathbb{Q}[X]$ (polynomial ring in one variable X). Show that V^* is not isomorphic to V.

(18*) Let f be a non-constant entire function with $f(0) = 0$. Let u and v be the real and imaginary parts of f respectively. Let $R > 0$ and

$$B = \sup\{u(z) : |z| = R\}.$$

(A) (2 marks) Show that $B > 0$.

(B) (2 marks) Consider the function

$$F(z) := \frac{f(z)}{z(2B - f(z))}.$$

Show that F is analytic on the open ball with radius R and continuous on the boundary $\{z : |z| = R\}$.

(C) (3 marks) Show that $\sup\{|F(z)| : |z| = R\} \leq \frac{1}{R}$.

(D) (3 marks) Show that

$$\sup\left\{|f(z)| : |z| = \frac{R}{2}\right\} \leq 2B.$$

(19*) Let $U(n)$ be the group of $n \times n$ unitary complex matrices. Let $P \subset U(n)$ be the set of all finite order elements of $U(n)$, that is, $P = \{X \in U(n) \mid X^m = 1 \text{ for some } m \geq 1\}$. Show that P is dense in $U(n)$.

(20*) Let A be a non-trivial subgroup of $\mathbb{R}$ generated by finitely many elements. Let r be a real number such that $x \longrightarrow rx$ is an automorphism of A. Show that r and r^{-1} are zeros of monic polynomials with integer coefficients.