CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
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Part A

(1) A, B, C.
(2) B, C, D.
(3) D.
(4) A, C, D.
(5) B, D.
(6) B, D.
(7) A, C, D.
(8) A.
(9) A, C, D.
(10) C, D.

Part B

(11) For $0 < x < 1$, write $x^{-x} = e^{\ln x^{-x}} = \sum_{n=0}^{\infty} 1/n!\ (-x\ln x)^n$. It converges pointwise. Since the terms in the summation are positive, by Dini's theorem, the convergence is uniform.

Change order of the summation and the integral using uniform convergence. Now integrate each term as follows to get the result.

$$\int_0^1 (-x\ln x)^n dx = \int_0^{\infty} u^n e^{-(n+1)u} du = \frac{1}{(n+1)^{n+1}} \int_0^{\infty} v^n e^{-v} dv = n! \frac{1}{(n+1)^{n+1}}.$$

(First equality by substitution $u = -\ln x$, second by $(n+1)u = v$, and then by repeated integration by parts.)

(12) (A) If PQ were a group, then, its order would be $|P||Q|/|P \cap Q|$.
Note that $P \cap Q \subset Q$. If $P \cap Q = Q$, then $Q \subset P$, a contradiction. If $P \cap Q \not\subset Q$, then $|Q|/|P \cap Q| = p^e$ for some $e \geq 1$, so PQ is a p-group bigger than P, a contradiction.

(B) The condition on Γ implies that all Sylow subgroups are normal and hence unique.
Write $|\Gamma| = p_1^{e_1} \cdots p_r^{e_r}$ with $e_i > 0$ for each $1 \leq i \leq r$. Write $|H| = p_1^{f_1} \cdots p_r^{f_r}$ with $f_i \geq 0$ for each $1 \leq i \leq r$. For $1 \leq i \leq r$, let P_i be the Sylow p_i-subgroup of Γ. Then $P_i \cap H = \langle 1 \rangle$ or is a p_i-subgroup of H. Conversely, if $f_i > 0$, then every p_i-subgroup of H is inside P_i, so is inside $P_i \cap H$. Hence if $f_i > 0$, then $P_i \cap H$ is the unique Sylow p_i-subgroup of H.
Now consider the group homomorphism

$$\prod_i (P_i \cap H) \longrightarrow H, (g_1, \ldots, g_r) \mapsto g_1 \cdots g_r.$$

(This indeed is a group homomorphism.) The orders of the groups on the two sides are the same, so it suffices to show that the homomorphism is injective. Suppose that $g_1 \cdots g_r = 1$. Then $g_1 = (g_2 \cdots g_r)^{-1} \in P_1 \cap P_2 \cdots P_r = \langle 1 \rangle$, so $g_1 = 1$. Repeatedly doing this, we see that $g_i = 1$ for each i.

(13) (A) Write $f = aX^n + g$ where g is a homogeneous polynomial of degree n that does have X^n as its term. Then $f(p) = a$. Moreover,

$$\frac{\partial f}{\partial X} = anX^{n-1} + \frac{\partial g}{\partial X}$$

so

$$\frac{\partial f}{\partial X}(p) = an.$$

Hence $a = an$, so $a = 0$. Hence $\dim V/W = 1$.

(B) Let V be the subspace of $\mathbb{R}^9$ spanned by $e_2, e_3, \ldots, e_9$ and W the subspace spanned by $e_2, e_3, \ldots, e_8$. It is clear that $T(V) = W$. In greater detail, by the formulas defining T, $Te_i \in W$ for $i = 2, \ldots, 9$, whence $T(V) \subset W$. Since the elements spanning W occur in $T(V)$, we get $T(V) = W$. Since $\mathbb{R}^9$ is spanned by V and e_1, it follows that $T(\mathbb{R}^9)$ is spanned by $T(V)$ and Te_1. In other words $T(\mathbb{R}^9)$ is the linear span of W and $e_1+e_3+e_8$. Since $e_1 \notin W$, it follows that $Te_1 = e_1+e_3+e_8 \notin W$. Thus $W+\mathbb{R}Te_1 = W \oplus \mathbb{R}Te_1$, whence $\dim T(\mathbb{R}^9) = \dim W + 1 = 7+1 = 8$. The nullity of T is therefore 1 by the rank-nullity theorem.

(14) (A) Can be checked directly.

(B) I is a prime ideal. In fact, $I = \ker \phi$ where $\phi : S \to R$ is the surjective ring map $\phi(f(x)) = f(0)$.

(C) I is a non-zero prime ideal that is not maximal. Hence S is not a PID.

(15) (A) We have $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [0,1]$. So the double integral

$$\int\int_{[0,1]\times[0,1]} (f(x) - f(y))(g(x) - g(y))\, dxdy \geq 0.$$

Iterating and expanding we get the inequality.

(B) By Rolle's Theorem, there exists $y_1 \in (0,1)$ such that $f'(y_1) = 0$. Since $f'(0) = 0$ (by hypothesis), we have, once again by Rolle's Theorem, an element $y_2 \in (0, y_1)$ such that $f^{(2)}(y_2) = 0$. If $n = 1$, we are done. Assume $n > 1$ and assume, by way of induction, that there exists $y \in (0, 1)$ such that $f^{(n)}(y) = 0$. Since $f^{(n)}(0) = 0$, one more application of Rolle's Theorem gives us an $x \in (0, y) \subset (0,1)$ such that $f^{(n+1)}(x) = 0$.

(16) (A) For $t \in \mathbb{R}$ we have, by squaring both sides of the given inequality of non-negative numbers,

$$(\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y}) \geq \mathbf{x} \cdot \mathbf{x}$$

whence

$$||\mathbf{x}||^2 + 2t(\mathbf{x} \cdot \mathbf{y}) + t^2||\mathbf{y}||^2 \geq ||\mathbf{x}||^2$$

which means

$$2t(\mathbf{x} \cdot \mathbf{y}) + t^2||\mathbf{y}|^2 \geq 0. \qquad (*)$$

For $t > 0$ this yields $2\mathbf{x} \cdot \mathbf{y} \geq -t||\mathbf{y}||^2$, and letting $t \longrightarrow 0^+$, we get $\mathbf{x} \cdot \mathbf{y} \geq 0$. For $t < 0$, $(*)$ implies $2\mathbf{x} \cdot \mathbf{y} \leq -t||\mathbf{y}||^2$, whence, letting $t \longrightarrow 0^-$, we get $\mathbf{x} \cdot \mathbf{y} \leq 0$. This gives the result.

(B) Parametrise the circle as $\gamma : [0,1] \longrightarrow \mathbb{R}^2$,

$$\gamma(t) = (\gamma_x(t), \gamma_y(t)) = (\cos 2\pi t, \sin 2\pi t)$$

Then, by definition, the required line integral is

$$\int_0^1 \{v_x(\cos 2\pi t, \sin 2\pi t))\frac{d\gamma_x(t)}{dt} + v_y(\cos 2\pi t, \sin 2\pi t)\frac{d\gamma_y(t)}{dt}\}dt$$
$$= \int_0^1 2\pi\{-(\cos 2\pi t - \sin 2\pi t)\sin 2\pi t + (\sin 2\pi t + \cos 2\pi t)\cos 2\pi t$$
$$= \int_0^1 2\pi\{\sin^2 2\pi t) + \cos^2 2\pi t\}dt = 2\pi$$

On the other hand, if $\vec{v}$ were a gradient, this line integral would vanish, so there is no function such that $\vec{v} = grad\, f$.

(17*) $\{1, X, X^2, \ldots, \}$ is a $\mathbb{Q}$-basis of V. Every $\mathbb{Q}$-linear map from V to $\mathbb{Q}$ is determined by what it does to the above basis. I.e., V^* as a set is the same as the set of functions from $\mathbb{N}$ to $\mathbb{Q}$. Hence V^* is an uncountable set, and, therefore, it cannot have a countable basis over $\mathbb{Q}$. Hence V^* is not isomorphic to V.

(18*) (A) By open mapping theorem, a nhd around zero will go to a nhd around zero and hence maximum modulus for U ensures positivity.

(B) The real part of f is at most B, hence real part of $2B - f$ does not vanish on the closed ball.

(C) On the boundary, $|2B - f(z)|^2 = (2B - U)^2 + V^2 \geq U^2 + V^2 = |f|^2$.

(D) Applying maximum modulus to note that for any w in the interior,

$$|\frac{f(w)}{w(2B-f(w))}| \leq \frac{1}{R}$$
$$|f(w)| \leq \frac{|w|}{R}(2B+|f(w)|)$$
$$|f(w)| \leq \frac{2|w|}{R-|w|}B$$

For $|w| = R/2$, we have $|f(w)| \leq 2B$.

(19*) When $n = 1$, the statement is valid since $U(1)$ is the group of unit complex numbers and P equals set μ of all the roots of unity $\{e^{2\pi i t} \mid t \in \mathbb{Q}\}$.

Now consider any $n \geq 1$. The diagonal subgroup $T \subset U(n)$ equals $U(1) \times \cdots \times U(1)$ and so $P \cap T = \mu \times \cdots \times \mu$ (n factors) is dense in T. Now let $X \in U(n)$ and let $V \subset X$ be any open neighbourhood of X. There exists a unitary matrix A such that $AXA^{-1} = AX\bar{A}^t \in U(n)$ is diagonal. Choose such an A. Now the set $W := AV\bar{A}^t = \{AY\bar{A}^t \mid Y \in V\}$ is an open subset of $U(n)$. Since $W \cap T$ is a non-empty open set in T and since $P \cap T$ is dense in T, there exists a finite order element $B \in W \cap T$. It follows that $A^{-1}BA \in V$ has finite order.

(20*) Since $A \subset \mathbb{R}$, A has no nontrivial finite order element. Since A is also a finitely generated abelian group, it is isomorphic to $\mathbb{Z}^n$ for some $n \geq 1$.

Let $\rho : A \longrightarrow A$ be the endomorphism of A defined by multiplication by r. Thus $\rho(a) = ra$ and $\rho^k(a) = r^k a$ for all $a \in A$ and $k \geq 1$. The matrix of ρ with respect to a basis of A has integer entries and with determinant equal to $= \pm 1$.

Note that $\rho^k(a) = r^k a$ for all $a \in A$. Suppose that $P(T) = T^n + c_1 T^{n-1} + \cdots + c_n$ be the characteristic polynomial of ρ. Then $0 = P(\rho)a = P(r)\cdot a$ for all $a \in A$. Choosing $a \in A$ to be non-zero, we see that $P(r) = 0$. Since $P(T)$ is monic with integer coefficients, r is an algebraic integer. Since the constant term of $P(T)$ equals $(-1)^n \det(\rho) = \pm 1$, r^{-1} is also an algebraic integer. (Or apply the same argument to r^{-1} to arrive at the same conclusion.)