CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
22nd May 2022

PART A

(1) A, C.
(2) D.
(3) A.
(4) A.
(5) B, C, D.
(6) B, C.
(7) A, D.
(8) B, C, D .
(9) B, C.
(10) B, C, D.

Part B

(11) (A) Multiplying by the transpose, we get

$$3 = \begin{bmatrix}1 & 1 & 1\end{bmatrix} A^t A \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix} \lambda^2 \begin{bmatrix}1\\1\\1\end{bmatrix} = 3\lambda^2$$

so $\lambda = \pm 1$.

(B) Similarly,

$$\lambda(x+y+z) = \begin{bmatrix}1 & 1 & 1\end{bmatrix} \lambda \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix} A^t A \begin{bmatrix}1\\-1\\0\end{bmatrix} = 0$$

so $x+y+z=0$ since $\lambda \neq 0$.

(12) The ratio above can be expressed as

$$\frac{\frac{1}{n}\sum_{j=1}^{n}(\frac{j}{n})^a}{(1+\frac{1}{n})^{a-1}\cdot\frac{1}{n}\sum_{j=1}^{n}(a+\frac{j}{n})}.$$

In the limit, the numerator becomes the integral $\int_0^1 x^a dx$. Likewise, the denominator tends to $\int_0^1 (a+x)dx$. Setting $S(a) = 1/60$ gives the quadratic equation $2a^2 + 3a - 119 = 0$. So the answer is $-3/2$.

(13) By way of contradiction, let $\alpha \in V \setminus f(U)$. Find a countable collection of nested closed balls $\{C_n\}$ around α whose intersection is $\{\alpha\}$. Then

$$\cap_n f^{-1}(C_n) = \varnothing.$$

But then $f^{-1}(C_n) = \phi$ for some n and hence the interior of C_n contains α and is in $V \setminus f(U)$. So $V \setminus f(U)$ is open, it is also closed, contradicting the hypothesis that V is connected. Thus $f(U) = V$.

(14) (A) By way of contradiction, assume that p and q are distinct prime numbers that divide $|G|$. Then G has subgroups of orders p and q, so such an N cannot exist. Hence G is a finite p-group for some p.

(B) Hence the centre $Z(G) \neq \{1_G\}$. Since $Z(G)$ is abelian, and a normal subgroup of G and N a subgroup of $Z(G)$, it follows that N is a normal subgroup of G.

(15) Chose $\alpha \in \mathbb{C}$ such that $\Re(\alpha) < 0$ and $\Re(\alpha f'(1)) > 0$. This is possible if $f'(1) \notin \mathbb{R}$. Since $f(D) \subset D$, for small $t > 0$

$$\Re\frac{f(1+t\alpha)-1}{t} < 0$$

and hence as $t \longrightarrow 0^+$, $\Re(\alpha f'(1)) \leq 0$.

For the second part, note that by Schwartz lemma and $0 < t < 1$, $|f(t)| \leq t$ and hence

$$\frac{|f(t)-1|}{1-t} \geq \frac{1-|f(t)|}{1-t} \geq 1$$

and as $t \longrightarrow 1$, we have the inequality.

(16) Since every finite extension of a finite field is Galois, F is infinite By way of contradiction, assume that $V = \bigcup_{i=1}^r V_i$. Pick $x \neq 0 \in V_1$ and $y \in V \setminus V_1$. Then there exists $j > 1$ such that $x + \alpha y \in V_j$ for infinitely many $\alpha \in F^\times$. Hence $y \in V_j$ and, so, $x \in V_j$. Therefore $V = \bigcup_{i=2}^r V_i$. Repeating this argument, we get a contradiction.

(17*) Fibre is

$$\mathbb{C}[X,Y]/(XY-1, X+\alpha Y-\beta).$$

If $\alpha=\beta=0$, then it is the zero ring since $(XY-1,X)=\mathbb{C}[X,Y]$. Otherwise,

$$\mathbb{C}[X,Y]/(XY-1,X+\alpha Y-\beta)\simeq \mathbb{C}[X]/(-(\alpha Y-\beta)Y-1)\simeq \mathbb{C}[Y]/((\alpha Y-\beta)Y+1)$$

The equation $\alpha Y^2-\beta Y+1$ has at least one solution; it has exactly one solution if and only if $\beta^2=4\alpha$. Hence for $n=0$, the answer is $\{(0,0)\}$. For each (α,β) with $\beta^2=4\alpha$, the fibre has exactly one prime ideal; for all other values of (α,β), the fibre has exactly two prime ideals.

(18*) (A) Consider the projection maps $\pi_i : Q \longrightarrow \mathbb{R}$, $(y_1,\dots)\mapsto y_i$. Then

$$Q_2=\bigcap_{i\geq 1}\pi_i^{-1}([0,\frac{1}{n}])$$

so it is a closed subset of Q. Since Q is compact by Tychonoff's theorem, Q_2 is compact.

(B) The maps $(y_1,\dots)\mapsto iy_i$ is continuous for each i, so D is continuous. It is bijective. Since Q is Hausdorff and Q_2 compact, D is continuous.

(C) Take $L=\ell_2$ and $\mathbf{a}=(1,\frac{1}{2},\dots,\frac{1}{n},\dots)$. (Note that $Q_2\subseteq\ell_2$.)

(D) D is a homeomorphism and $S\circ D$ is continuous.

(19*) (A) is false since the polynomial $X^{12}-p$ has two real roots.

(B) is a consequence of the fact that any degree 11 polynomial has a real root.

(C) If the group $\mathrm{Aut}(E)$ of all field automorphisms of E has odd order, then every embedding $E\longrightarrow\mathbb{C}$ is real. note that if σ is a complex embedding, then complex conjugation yields a nontrivial automorphism E of order 2.

(20*) (A) $d_X(a,-a)=2R$. Hence $2R/\lambda\leq d_Y(f(a),f(-a))\leq L_R+2$, so $L_R\geq 2R/\lambda-2$. $L_R=d_Y(f(a),f(b))\leq\lambda d_X(a,b)$. Hence $d_X(a,b)\geq L_R/\lambda\geq 2R/\lambda^2-2/\lambda$.

(B) Note that $\bar{f}(C_1)=\bar{f}(C_2)=\bar{f}(C_R)$. Hence there exist such x_i as asserted. Since $\{f(x_1),f(x_2)\}\subseteq\mathbb{S}^1\times\{\frac{f(a)+f(b)}{2}\}$, it follows that $d_Y(f(x_1),f(x_2))\leq 2$.

(C) $d_Y(f(x_1),f(a))\geq L_R/2$, so $d_X(x_1,a)\geq L_R/2\lambda\geq(R-\lambda)/\lambda^2$. Similarly $d_X(x_2,a)\geq L_R/2\lambda\geq(R-\lambda)/\lambda^2$. Choose $a_i\in C_i$ be such that $d_X(a,a_i)=(R-\lambda)/\lambda^2$; Elementary trigonometric arguments show that for all $R\gg 0$, $d_X(a_1,a_2)>2\lambda$. Hence $d_X(x_1,x_2)>2\lambda$. Therefore $d_Y(f(x_1),f(x_2))>2$.

(D) There does not exist any bilipschitz map $X\longrightarrow Y$.