CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
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Part A

(1) A, B, D.
(2) C.
(3) B, C.
(4) A, B.
(5) A, B.
(6) B, D.
(7) B.
(8) B, C, D.
(9) A, C.
(10) A, B, D.

Part B

(11) Write Z for the centre of G. If $Z \not\subset M$, then $MZ > M$ and by maximality $MZ = G$. But then M is a normal subgroup of G, a contradiction. Conversely $\langle M, N\rangle = G$ by maximality. Also since M and N are abelian, we have

$$G = \langle M, N\rangle \subset C_G(M \cap N)$$

and hence $(M \cap N) \subset Z$.

(12) $f(D) \cap D$ is open in D by Inverse mapping theorem. $f(D) \cap D$ is also closed in D. Let $z_n = f(x_n)$ be a sequence in $f(D) \cap D$ converging to $\alpha \in D$ with $x_n \in D$. Then $\alpha = f(\beta)$ for some $\beta \in \overline{D}$. But the hypothesis ensures that $\beta \in D$. $f(D) \cap D$ is non-empty as 0 belongs to this. So $f(D) \cap D = D$.

(13) Union of connected sets having a point in common is connected. So A is connected. Closure of a connected set is connected. So $\overline{A}$ is connected and, hence, $\overline{A} \in \mathcal{S}$ so $\overline{A} \subseteq A$.

(14) Without loss of generality, we may assume that Image$(f) \subseteq (0, \infty)$.

Let $\epsilon = 1$. There exists a positive integer N such that for all $x, x' \in [1, \infty)$ with $|x - x'| < \frac{1}{N}$, $|f(x) - f(x')| < 1$. Hence $|f(n+1) - f(n)| < N$ for each positive integer n. Therefore $f(n+1) < f(1) + nN$ for each positive integer n. Hence

$$\sum_{n\geq 1} \frac{1}{f(n)} \geq \sum_{n\geq 0} \frac{1}{f(1) + nN}$$

which is a divergent series.

(15) On $[0, 1]$ it is a continuous function. Need only show integral from 1 to infinity is finite. Change variable $x = y^{100}$, $dx = 100y^{99}dy$ the integral becomes $100 \int_1^\infty y^c e^{-y} dy$ for some c and if you take an integer $N > c$ the integral is bounded by $\int_1^\infty y^N e^{-y} dy$ which is finite.

(16) False in both cases. Take $F = \mathbb{F}_3$ the finite field of order 3 and $I_1 = (x^2 + 1)$ and $I_2 = (x^2 + 2x + 2)$. Both these are maximal ideals (check that the generating polynomials are irreducible since they have no roots) and hence the quotients $R/I_1, R/I_2$ are both finite fields of order 9 and they isomorphic. But $I_1 \neq I_2$. If $F = \mathbb{R}$, then for every irreducible quadratic polynomial $f(X) \in R$, $R/(f(X)) \simeq \mathbb{C}$.

(17*) (A) Let v_1, v_2 be the columns of A; they form a basis of $\mathbb{R}^2$. Let $u_1 = v_1/|v_1|$, $a = (u_1 \cdot v_2)$, $v_2' = v_2 - au_1$, $u_2 = v_2'/|v_2'|$. Since u_1 and u_2 form an orthonormal basis of $\mathbb{R}^2$, define A_o to be the matrix with columns u_1, u_2. Define A_b to be the inverse of

$$\begin{bmatrix} \frac{1}{|v_1|} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{|v_2'|} \end{bmatrix}.$$

Then $A = A_o A_b$.

(B) Suppose $A'A'' = A_1'A_1''$, with $A', A_1' \in \mathrm{O}(2, \mathbb{R})$ and $A'', A_1'' \in \mathrm{B}_+(2, \mathbb{R})$. Then $A''(A_1'')^{-1} \in \mathrm{O}(2, \mathbb{R})$. Note that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix} \in \mathrm{O}(2, \mathbb{R})$$

if and only if

$$\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^{-1}.$$

Therefore $A'' = A_1''$, and, hence, $A' = A_1''$.

(C) By above, there is a well-defined function

$$\psi : \mathrm{GL}(2,\mathbb{R}) \longrightarrow \mathrm{O}(2,\mathbb{R}) \times \mathrm{B}_+(2,\mathbb{R}) \qquad A \mapsto (A_o, A_b)$$

that is the inverse of ϕ.
We first show that the map $A \mapsto (A_b)^{-1}$ is continuous. From the description of $(A_b)^{-1}$ given above, we see that its entries are rational functions of the entries of A, with non-zero denominators (since $A \in \mathrm{GL}(2,\mathbb{R})$. Hence the map $A \mapsto (A_b)^{-1}$ is continuous. Hence the maps $A \longrightarrow A_o = A(A_b)^{-1}$ and $A \longrightarrow A_b$ are continuous. Therefore ψ is continuous.
ϕ is continuous: The entries of $A'A''$ are polynomial functions of the entries of A' and of A''.

(18*) ϕ satisfies $T^{p^3} - 1 = 0$. Hence the only eigenvalue of ϕ is 1. Let v_1 be an eigenvector for the eigenvalue 1. Then ϕ induces an invertible linear transformation of $V/\langle v_1\rangle$; proceed by induction on dimension.

(19*) (A) First, $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$ since the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$. Next, $[\mathbb{Q}(\sqrt[5]{2}) : \mathbb{Q}] = 5$ since $x^5 - 2$ is irreducible over $\mathbb{Q}$. For both use the Eisenstein criterion. So $[K : \mathbb{Q}]$ is divisible by both 4 and 5. On the other hand, we have $[K : \mathbb{Q}(\zeta_5)] \leq 5$. So we conclude $[K : \mathbb{Q}] = 20$.

(B) If $\imath \in \mathbb{Q}(\zeta_5)$, then $\alpha := \imath\zeta_5 \in \mathbb{Q}(\zeta_5)$. Note that $\alpha^{20} = 1$ and $\alpha^n \neq 1$ for $1 \leq n \leq 19$. So α is a primitive 20th root of unity and it is contained in $\mathbb{Q}(\zeta_5)$. But we know that $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$ and by the fact given in the hint, we have $[\mathbb{Q}(\alpha) : \mathbb{Q}] > 4$. So $\imath \notin \mathbb{Q}(\zeta_5)$.

(C) Suppose $\imath \in K$. Consider the tower $\mathbb{Q} \subset \mathbb{Q}(\zeta_5) \subset \mathbb{Q}(\zeta_5, \imath) \subset K$. The degree of the first extension is 4; the degree of the second extension is 2 (since $\imath \notin \mathbb{Q}(\zeta_5)$). So $[K : \mathbb{Q}]$ is divisible by 8. This is not possible since $[K : \mathbb{Q}] = 20$.

(20*) (A) Write $f(z) = \sum_{n\geq 0} a_n z^n$. Then

$$\begin{aligned} f(z) - 2zf(z) - z^2 f(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \\ &\quad - 2a_0 z - 2a_1 z^2 - 2a_2 z^3 + \cdots \\ &\quad - a_0 z^2 - a_1 z^3 - a_2 z^4 + \cdots \\ &= a_0 + (a_1 - 2a_0) z \end{aligned}$$

Let

$$g(z) = \frac{a_0 + (a_1 - 2a_0)z}{1 - 2z - z^2}$$

This is analytic in a neighbourhood of the origin. Let $\sum_{n\geq 0} b_n z^n$ be the Taylor series expansion of $g(z)$ around 0. Then $(1 - 2z - z^2)f(z) = (1 - 2z - z^2)g(z)$, from which we see that $a_n = b_n$ for every n.

(B) Note that

$$g(z) = \frac{1}{1 - 2z - z^2}.$$

Let γ_1, γ_2 be the roots of $1 - 2z - z^2$. Note that $\gamma_1 \neq \gamma_2$. Then there exist β_1, β_2 such that

$$g(z) = \frac{\beta_1}{z - \gamma_1} + \frac{\beta_2}{z - \gamma_2}.$$

Therefore

$$g^{(n)}(z) = \frac{(-1)^n n! \beta_1}{(z - \gamma_1)^{n+1}} + \frac{(-1)^n n! \beta_2}{(z - \gamma_2)^{n+1}}.$$

Hence

$$a_n = -\frac{\beta_1}{\gamma_1^{n+1}} - \frac{\beta_2}{\gamma_2^{n+1}}.$$

Now replace β_i by $-\beta_i$ and γ_i by $\frac{1}{\gamma_i}$.