CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
4th October 2020

Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.**

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, of non-negative real numbers, of positive real numbers, and of complex numbers. For a prime power q, $\mathbb{F}_q$ is the field with q elements. For a field F, $M_n(F)$ stands for the set of $n \times n$ matrices over F. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}$ are taken with the euclidean topology.

Part A

Instructions: Each of the questions 1–9 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let G be a group and N be a proper normal subgroup. Pick the true statement(s) from below.
  (A) If N and the quotient G/N is finite, then G is finite.
  (B) If the complement $G \setminus N$ of N in G is finite, then G is finite.
  (C) If both N and the quotient G/N are cyclic, then G is cyclic.
  (D) G is isomorphic to $N \times G/N$.

(2) Let R denote the ring of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, where addition and multiplication are given, respectively, by $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$ for every $f, g \in R$ and $x \in \mathbb{R}$. A *zero-divisor* in R is a non-zero $f \in R$ such that $fg = 0$ for some non-zero $g \in R$. Pick the true statement(s) from below:
  (A) R has zero-divisors.
  (B) If f is a zero-divisor, then $f^2 = 0$.
  (C) If f is a non-constant function and $f^{-1}(0)$ contains a non-empty open set, then f is a zero-divisor.
  (D) R is an integral domain.

(3) Let $U = \{(x, y) \in \mathbb{R}^2 \mid x < y^2 < 4\}$ and $V = \{(x, y) \in \mathbb{R}^2 \mid 0 < xy < 4\}$, both taken with the subspace topology from $\mathbb{R}^2$. Which of the following statement(s) is/are true?
  (A) There exists a non-constant continuous map $V \longrightarrow \mathbb{R}$ whose image is not an interval.
  (B) Image of U under any continuous map $U \longrightarrow \mathbb{R}$ is bounded.
  (C) There exists an $\epsilon > 0$ such that given any $p \in V$ the open ball $B_\epsilon(p)$ with centre p and radius ϵ is contained in V.
  (D) If C is a closed subset of $\mathbb{R}^2$ which is contained in U, then C is compact.

(4) Let A and B be 5×5 real matrices with $A^2 = B^2$. Which of the following statements is/are correct?
  (A) Either $A = B$ or $A = -B$.
  (B) A and B have the same eigen spaces.
  (C) A and B have the same eigen values.

(D) $A^{13}B^3 = A^3B^{13}$.

(5) Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y) = \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2+y^2}$$

for $y \neq 0$ and $f(x,0) = 0$. (The square root is chosen to be non-negative). Pick the correct statement(s) from below:

(A) f is continuous at $(0,0)$.
(B) f is an open map.
(C) f is differentiable at $(0,0)$.
(D) f is a bounded function.

(6) Which of the following is/are true for a series of real numbers $\sum a_n$?

(A) If $\sum a_n$ converges then $\sum a_n^2$ converges;
(B) If $\sum a_n^2$ converges then $\sum a_n$ converges;
(C) if $\sum a_n^2$ converges then $\sum \frac{1}{n} a_n$ converges;
(D) If $\sum |a_n|$ converges then $\sum \frac{1}{n} a_n$ converges;

(7) Which of the following functions are uniformly continuous on $\mathbb{R}$?

(A) $f(x) = x$;
(B) $f(x) = x^2$;
(C) $f(x) = (\sin x)^2$;
(D) $f(x) = e^{-|x|}$.

(8) Let U and V be non-empty open connected subsets of $\mathbb{C}$ and $f : U \longrightarrow V$ a analytic function. Which of the following statement(s) is/are true?

(A) $f'(z) \neq 0$ for every $z \in U$.
(B) If f is bijective, then $f'(z) \neq 0$ for every $z \in U$.
(C) If $f'(z) \neq 0$ for every $z \in U$, then f is bijective.
(D) If $f'(z) \neq 0$ for every $z \in U$, then f is injective.

(9) Let U denote the unit open disc centred at 0. Let $f : U \setminus \{0\} \longrightarrow \mathbb{C}$ be an analytic function. Assume that $\lim_{z \longrightarrow 0} zf(z) = 0$.

(A) $\lim_{z \longrightarrow 0} |f(z)|$ exists and is in $\mathbb{R}$.
(B) f has a pole of order 1 at 0.
(C) $zf(z)$ has a zero of order 1 at 0.
(D) There exists an analytic function $g : U \longrightarrow \mathbb{C}$ such that $g(z) = f(z)$ for every $z \in U \setminus \{0\}$.

Instructions: The answers to Question 10 is an integer. Please write the answer in decimal form in the attached bubble-sheet. The question is worth four (4) marks.

(10) Let $f(x) = x^2 + ax + b \in \mathbb{F}_3[X]$. What is the number of non-isomorphic quotient rings $\mathbb{F}_3[X]/(f(X))$?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

(11) Let (X,d) be a compact metric space. For $x \in X$ and $\epsilon > 0$, define $B_\epsilon(x) := \{y \in X \mid d(x,y) < \epsilon\}$. For $C \subseteq X$ and $\epsilon > 0$, define $B_\epsilon(C) := \cup_{x\in C} B_\epsilon(x)$. Let $\mathcal{K}$ be the set of non-empty compact subsets of X. For $C, C' \in \mathcal{K}$, define $\delta(C,C') = \inf\{\epsilon \mid C \subseteq B_\epsilon(C')$ and; $C' \subseteq B_\epsilon(C)\}$. Show that $(\mathcal{K}, \delta)$ is a compact metric space.

(12) Let f be a non-constant entire function with $f(z) \neq 0$ for all $z \in \mathbb{C}$. Consider the set $U = \{z : |f(z)| < 1\}$. Show that all connected components of U are unbounded.

(13) Let $F \subseteq \mathbb{R}^3$ be a non-empty finite set, and $X = \mathbb{R}^3 \setminus F$, taken with the subspace topology of $\mathbb{R}^3$. Show that X is homeomorphic to a complete metric space. (Hint: Look for a suitable continuous function from X to $\mathbb{R}$.)

(14) Show that there is no differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(0) = 1$ and $f'(x) \geq (f(x))^2$ for every $x \in \mathbb{R}$.

(15) Let $a_1, \ldots, a_n$ be distinct complex numbers. Show that the functions $e^{a_1 z}, \ldots, a^{a_n z}$ are linearly independent over $\mathbb{C}$.

(16) The *Frattini subgroup* of a finite group G is the intersection of all its proper maximal subgroups. Let p be a prime number. Show that the Frattini subgroup of $\mathbb{Z}/p^n$, $n \geq 2$, is generated by p.

(17*) Let $M \in M_n(\mathbb{C})$. Show that M is diagonalizable if and only if for every polynomial $P(X) \in \mathbb{C}[X]$ such that $P(M)$ is nilpotent, $P(M) = 0$.

(18*) X is said to have the *universal extension property* if for every normal space Y and every closed subset $A \subset Y$ and every continuous function $f : A \longrightarrow X$, f extends to a continuous function from Y to X. You may assume, without proof, that $\mathbb{R}^2$ has the universal extension property.

(A) Prove or find a counterexample: If X has the universal extension property, then X is connected.

(B) Give an example (with justification) of a compact subset X of $\mathbb{R}^2$ that does not have the universal extension property.

(C) Let $X = \{(x, \sin x) \mid x \in \mathbb{R}\}$. Then show that X has the universal extension property.

(19*) Let p be a prime number and q a power of p. Let K be an algebraic closure of $\mathbb{F}_q$. Say that a polynomial $f(X) \in K[X]$ is a *q-polynomial* if it is of the form

$$f(X) = \sum_{i=0}^{n} a_i X^{q^i}$$

Let $f(X)$ be a q-polynomial of degree q^n, with $a_0 \neq 0$. Show that the set of zeros of $f(X)$ is an n-dimensional vector-space over $\mathbb{F}_q$.

(20*) Let $a_n, n \geq 1$ be a sequence of real numbers. If $a_n \to a$, show that

$$b_n = \frac{a_1 + 2a_2 + 3a_3 + \cdots + na_n}{n^2} \to \frac{a}{2}.$$