CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
4th October 2020

Part A

(1) A, B.
(2) A, C.
(3) A.
(4) D.
(5) A.
(6) C, D.
(7) A, C, D.
(8) B.
(9) A, D.
(10) 3.

Part B

(11) It is easily checked that δ is a metric. Let C_k be a sequence of compact subsets of X. We must show that there is a convergent subsequence.

Given any $\epsilon > 0$, using there exists an $N = N(\epsilon)$ such that the *every* C_k is covered by (atmost) $N(\epsilon)$ many open balls of radius ϵ with centres in C_k. We do this first for X with $\epsilon/2$ balls and then take, for each such ball B, an ϵ-ball about a point in C_k in case the intersection $B \cap C_k$ is non-empty. We let ϵ vary over $1/n, n \in \mathbb{N}$. This way we get a finite set $F_{k,n} \subset C_k$ of centres of balls used to cover C_k. We may assume that $F_{k,n} \subset F_{k,n+1}$ by including balls of radius (n+1) around points of $F_{k,n}$. The cardinality of $F_{k,n}$ is at most $\sum_{m \leq n} N(1/m) =: N_n$ which is independent of k.) Then $\delta(F_{k,n}, C_k) < 1/n$. We list the points of $\cup_{n \geq 1} F_{k,n}$ in a sequence: $x_{k,1}, x_{k,2}, \ldots, x_{k,j}, \ldots$, viewed as kth row of a matrix whose rows and columns are labelled by $\mathbb{N}$. It is understood that in the above sequence we list members of $F_{k,i}$ before those of $F_{k,i+1}$ $\forall i \geq 1$ for each $k \geq 1$.

We consider the first column. Now $x_{k,1}, k \in \mathbb{N}$ is a sequence of points in X which has a convergent subsequence $x_{k_r,1} \in C_{k_r}, r \in N$. Set y_1 to be the limit of this subsequence. We will denote C_{k_1} as C_1^1.

We consider second column entries corresponding to the rows labelled by the subsequence k_r—that is the sequence $x_{k_r,2}$. This has a convergent subsequence, say with limit y_2. We will denote the first term of the corresponding subsequence by C_2^2.

Proceeding thus we obtain a sequence of points $y_1, \ldots, y_n, \ldots$. We let C be the closure of $\{y_k \mid k \geq 1\}$. Then C is compact.

Our claim is that the "diagonal" sequence C_k^k converges to C.

Let $n > 0$ be a positive integer. Given any $m \in \mathbb{N}$ we have $\delta(C_m, F_{m,n}) < 1/n$ and so $\delta(x_{m,r}, F_{m,n}) < 1/n$ for all r. So y_r is at a distance at most $1/n$ from $F_n := \{y_1, y_2, \ldots, y_m \mid m \leq N_n\}$ where $N_n = \sum_{l \leq n} N(1/l)$. So it follows that $\delta(C, F_n) \leq 1/n$.

If $C_k^k = C_r$, ($r = r(k)$ depends on k) for sufficiently large k we have $\delta(x_{r,j}, y_j) < 1/n$ for $j \leq N_n$. Hence we have $\delta(F_{r,n}, F_n) < 1/n$. So

$$\delta(C_r, C) \leq \delta(C_r, F_{r,n}) + \delta(F_{r,n}, F_n) + \delta(C, F_n) < 3/n.$$

(12) Let C be a component.Then C is open. If C is bounded, the value of $|f|$ on its boundary ∂C is 1 (by continuity, open mapping theorem and because ∂C is disjoint with C). Hence same for $1/|f|$. But this contradicts $|f(z)| < 1$ on C.

(13) Let $f : X \longrightarrow \mathbb{R}$ be given by $x \mapsto \frac{1}{\min\{d(x,p)|p\in F\}}$. It is continuous. The graph $\Gamma_f := \{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times \mathbb{R}$, hence a complete metric space. $X \mapsto \Gamma_f, x \mapsto (x, f(x))$ is a homeomorphism.

(14) Note that $f[0,\infty) \subset [0,\infty)$ as $f' \geq 0$. Integrating f'/f^2 on $[0, x]$ gives

$$1 - 1/f(x) \geq x$$

and hence f is unbounded in $[0, 1)$.

(15) Let $c_1, \dots, c_n \in \mathbb{C}$ be such that $\sum_i c_i e^{a_i z} = 0$. Differentiating this $n-1$ times, we see that $\sum_i c_i a_i^j e^{a_i z} = 0$ for every $0 \leq j \leq n-1$. Substitute $z = 0$ to get $\sum_i c_i a_i^j = 0$ for every $0 \leq j \leq n-1$. The Vandermonde matrix $(a_i^j)_{i,j}$ is invertible, so $c_i = 0$ for every i.

(16) contain $\overline{m}$ for any integer m with $\gcd(m, p^n) = 1$, so $H = \{\overline{jp} \mid 0 \leq j < p\}$.

(17*) If M is diagonalizable, then we may assume that M is a diagonal matrix. Then $P(M)$ is a diagonal matrix. Such a matrix is nilpotent if and only if it is zero. Conversely, let $\alpha_1, \dots, \alpha_m$ be the distinct eigenvalues of M. Let $P(X) = \prod_{i=1}^m (X - \alpha_i)$. Let $\mu(X)$ be the minimal polynomial of M. Since the roots of $\mu(X)$ are exactly $\alpha_1, \dots, \alpha_m$, there exists a positive integer r such that $(P(X))^r$ is divisible by $\mu(X)$. Hence $(P(M))^r = 0$. Therefore $P(M) = 0$, i.e., $P(X) = \mu(X)$. Therefore M is diagonalizable.

(18*) (1) True: In fact, X is path connected. We know the closed interval [0,1] is normal. Given two points $x, y \in X$, consider the function $f : \{0, 1\} \longrightarrow X$ defined by $f(0) = x, f(1) = y$. Then f is continuous and extends to all of $[0, 1]$, by the hypothesis on X.

(2) False: take a finite set of cardinality at least 2. Then it is compact and not connected. So it can't have universal extension property.

(3) Note that $X \subset \mathbb{R}^2$ is a retract: $r : \mathbb{R}^2 \longrightarrow X$ given by $f(x, y) = (x, \sin x)$ is a retraction. Since $\mathbb{R}^2$ has universal extension property, any map $A \longrightarrow \mathbb{R}^2$ can be extended to all of Y (for any given pair of a normal space Y and a closed subset $A \subset Y$). If we are given a function $f : A \longrightarrow X$, composing with the inclusion $X \longrightarrow \mathbb{R}^2$, we have an extension $g : Y \longrightarrow \mathbb{R}^2$. Composing this with r, we get the desired extension $Y \longrightarrow X$.

Note that it is not required to know about the language of retracts for (3). They will have to notice that a continuous map like r exists.

(19*) If $f(a) = f(b) = 0$ for some $a, b \in K$, then so $f(\lambda a + b) = 0$ for every $\lambda \in \mathbb{F}_q$. Hence the set of zeros is an $\mathbb{F}_q$-vector-space. It is n-dimensional since f is separable.

(20*) First $n(n+1)$ and n^2 are of the same order (the ratio converges to one). So you can as well take denominator as $n(n+1)$. Thus need to show

$$\frac{a_1 + 2a_2 + 3a_3 + \cdots + na_n}{n(n+1)/2} \to a.$$

Imitate blindly Cesaro theorem. Given $\epsilon > 0$, choose K so that $|a_n - a| < \epsilon/2$ for $n \geq K$. Then choose $N > K$ so that the finite sum

$$\frac{|a_1 - a| + 2|a_2 - a| + 3|a_3 - a| + \cdots + K|a_K - a|}{N(N+1)/2} < \epsilon/2.$$

If now $n > N$ then

$$|b_n - a| = |\frac{a_1 + 2a_2 + 3a_3 + \cdots + na_n}{n(n+1)/2} - a|$$

(Use $1 + \cdots + n = n(n+1)/2$ to distribute a to each term in numerator.)

$$\leq \frac{|a_1 - a| + 2|a_2 - a| + 3|a_3 - a| + \cdots + n|a_n - a|}{n(n+1)/2}$$

(split first K terms and the remaining.)

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$