CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2019

Instructions:

- Enter your *Admit Card Number* here: | M | – | | | | – | | | | |
- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

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Admit Card Number: **M** – ☐☐☐ – ☐☐☐☐

Bubble Sheet

Part A

1. (A) (B) (C) (D)
2. (A) (B) (C) (D)
3. (A) (B) (C) (D)
4. (A) (B) (C) (D)
5. (A) (B) (C) (D)
6. (A) (B) (C) (D)
7. (A) (B) (C) (D)
8. (A) (B) (C) (D)
9. (A) (B) (C) (D)
10. __________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*)**.**

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Number of correct answers in Part A:	
Marks in Part A:	

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Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.**

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, of non-negative real numbers, of positive real numbers, and of complex numbers. For a field F, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}$ are taken with the euclidean topology. When $M_n(\mathbb{R})$ is considered as a topological space, it is identified with $\mathbb{R}^{n^2}$.

Part A

Instructions: Each of the questions 1–9 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) For a field F, $F^{\times}$ denotes the multiplicative group $(F \setminus \{0\}, \times)$. Choose the correct statement(s) from below:
(A) Every finite subgroup of $\mathbb{R}^{\times}$ is cyclic;
(B) The order of every non-trivial finite subgroup of $\mathbb{R}^{\times}$ is a prime number;
(C) There are infinitely many non-isomorphic non-trivial finite subgroups of $\mathbb{R}^{\times}$;
(D) The order of every non-trivial finite subgroup of $\mathbb{C}^{\times}$ is a prime number.

(2) Let R be a commutative ring with 1 and I and J ideals of R. Choose the correct statement(s) from below:
(A) If I or J is maximal then $IJ = I \cap J$;
(B) If $IJ = I \cap J$, then I or J is maximal;
(C) If $IJ = I \cap J$, then $1 \in I + J$;
(D) If $1 \in I + J$ then $IJ = I \cap J$.

(3) Let (X, d) and (Y, ρ) be metric spaces and $f : X \longrightarrow Y$ a homeomorphism. Choose the correct statement(s) from below:
(A) If $B \subseteq Y$ is compact, then $f^{-1}(B)$ is compact;
(B) If $B \subseteq Y$ is bounded, then $f^{-1}(B)$ is bounded;
(C) If $B \subseteq Y$ is connected, then $f^{-1}(B)$ is connected;
(D) If $\{y_n\}$ is Cauchy in Y, then $\{f^{-1}(y_n)\}$ is Cauchy in X.

(4) Let $a, b \in \mathbb{R}$, and consider the $\mathbb{R}$-linear map $f : \mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto az + b\overline{z}$. Choose the correct statement(s) from below:
(A) f is onto (i.e., surjective) if $ab \neq 0$;
(B) f is one-one (i.e., injective) if $ab \neq 0$;
(C) f is onto if $a^2 \neq b^2$;
(D) if $a^2 = b^2$, f is not one-one.

(5) Let

$$f(x,y) = \begin{cases} \frac{x^3y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Choose the correct statement(s) from below:
(A) f is continuous on $\mathbb{R}^2$;
(B) f is continuous at every point of $\mathbb{R}^2 \setminus \{(0,0)\}$;
(C) f is differentiable at every point of $\mathbb{R}^2 \setminus \{(0,0)\}$;
(D) f is not differentiable at $(0,0)$.

(6) Let K be the smallest subfield of $\mathbb{C}$ containing all the roots of unity. Choose the correct statement(s) from below:
(A) $\mathbb{C}$ is algebraic over K;
(B) K has countably many elements;
(C) Irreducible polynomials in $K[X]$ do not have multiple roots;
(D) The characteristic of K is zero.

(7) The power series

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$$

equals
(A) $x^2 e^x$;
(B) xe^x;
(C) $(x^2 + x)e^x$;
(D) $(x^2 - x)e^x$;

(8) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice continuously differentiable. Suppose further that $f''(x) \geq 0$ for every $x \in \mathbb{R}$. Choose the correct statement(s) from below:
(A) f is bounded;
(B) f is constant;
(C) If f is bounded, then it is infinitely differentiable;
(D) $\int_0^x f(t)\mathrm{d}t$ is infinitely differentiable with respect to x.

(9) Let $f(z)$ be a power-series (with complex coefficients) centred at $0 \in \mathbb{C}$ and with a radius of convergence 2. Suppose that $f(0) = 0$. Choose the correct statement(s) from below:
(A) $f^{-1}(0) = \{0\}$;
(B) If f is a non-constant function on $\{|z| < 2\}$, then $f^{-1}(0) = \{0\}$;
(C) If f is a non-constant function, then for all $\zeta \in \mathbb{C}$ with sufficiently small $|\zeta|$, the equation $f(z) = \zeta$ has a solution;
(D)
$$\int_\gamma f^{(n)}(z)\mathrm{d}z = 0$$
for every $n \geq 1$, where γ is a unit circle centred at 0, oriented clockwise, and $f^{(n)}$ is the nth derivative of $f(z)$.

Instructions: The answer to Question 10 is an integer. You are required to write the answer in decimal form in the attached bubble-sheet. The question is worth four (4) marks.)

(10) Let z be a complex variable, and write $x = \Re(z)$ and $y = \Im(z)$ for the real and the imaginary parts, respectively. Let $f(z)$ be a complex polynomial. Let $R > 0$ be a real number and γ the circle in $\mathbb{C}$ of radius R and centre at 0, oriented in the counter-clockwise direction. What is the value of
$$\frac{1}{2\pi\imath R}\int_\gamma (\Re(f(z))\mathrm{d}x + \Im(f(z))\mathrm{d}y)$$

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Fix a non-negative integer d. Let

$$\mathcal{A}_d := \{A \subseteq \mathbb{C} : A \text{ is the zero-set of a polynomial of degree} \leq d \text{ in } \mathbb{C}[X]\}.$$

Let $\mathcal{T}$ be the coarsest topology on $\mathbb{C}$ in which A is closed for every $A \in \mathcal{A}_d$.

(A) Determine whether $\mathcal{T}$ is Hausdorff.

(B) Show that for every polynomial $f(X) \in \mathbb{C}[X]$, the function $\mathbb{C} \longrightarrow \mathbb{C}$ defined by $z \mapsto f(z)$ is continuous, where $\mathbb{C}$ (on both the sides) is given the topology $\mathcal{T}$.

(12) Let $a_n, n \geq 0$ be complex numbers such that $\lim_n a_n = 0$.

(A) Show that $F(z) := \sum_{n\geq 0} a_n z^n$ is a holomorphic function on $\{z \in \mathbb{C} : |z| < 1\}$.

(B) Let $G(z)$ be a meromorphic function on $\{z \in \mathbb{C} : |z| < 2\}$, with a pole at 1. Show that $G \neq F$ on $\{z \in \mathbb{C} : |z| < 1\}$. (Hint: consider the function $(1 - z)F(z)$ as $z \longrightarrow 1$.)

(13) Let $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be a function such that for every $x, y \in \mathbb{R}$, (i) $|x| = 0$ if and only if $x = 0$; (ii) $|x + y| \leq |x| + |y|$; (iii) $|xy| = |x||y|$. Show that the following are equivalent:

(A) The set $\{|n| : n \in \mathbb{Z}\}$ is bounded;

(B) $|x + y| \leq \max\{|x|, |y|\}$ for every $x, y \in \mathbb{R}$.

(14) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function. Show that the sequence

$$\left[\int_0^1 |f(x)|^n \mathrm{d}x\right]^{\frac{1}{n}}$$

is convergent.

(15) Let V be a subspace of the complex vector space $M_n(\mathbb{C})$. Suppose that every non-zero element of V is an invertible matrix. Show that $\dim_{\mathbb{C}} V \leq 1$.

(16) Let n be a positive integer such that every group of order n is cyclic. Show the following.

(A) For all prime numbers p, p^2 does not divide n.

(B) If p and q are prime divisors of n, then p does not divide $q-1$. (Hint: Consider 2×2 matrices

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$$

with $x, y \in \mathbb{Z}/q\mathbb{Z}$ and $x^p = 1$.)

(C) Show that $(n, \phi(n)) = 1$, where $\phi(n)$ is the number of integers m such that $1 \leq m \leq n$ with $\gcd(n, m) = 1$.

(17*) Let F be a field and $G = \mathrm{GL}_n(F)$. For $g \in G$, write $C_g = \{hgh^{-1} \mid h \in G\}$. Let $X = \{C_g \mid g \in G, \text{ the order of } g \text{ is } 2\}$. Determine $|X|$.

(18*) A *compactification* of a topological space X is a compact topological space Y which contains a dense subspace homeomorphic to X. Let $X = (0, 1]$, in the subspace topology of $\mathbb{R}$ and $f : X \longrightarrow \mathbb{R}, x \mapsto \sin \frac{1}{x}$. Show the following:

(A) $Y := [0, 1]$ is a compactification of X, but f does not extend to a continuous function $Y \longrightarrow \mathbb{R}$, i.e., there does not exist a continuous function $g : Y \longrightarrow \mathbb{R}$ such that $g|_X = f$.

(B) X is homeomorphic to the set $X_1 := \{(t, \sin \frac{1}{t}) \mid t \in X\} \subseteq \mathbb{R}^2$.

(C) The closure Y_1 of X_1 in $\mathbb{R}^2$ is a compactification of X

(D) f extends to a continuous function $Y_1 \longrightarrow \mathbb{R}$.

(19*) Let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial. Suppose that $\alpha \in \mathbb{C}$ and 3α are roots of f.

(A) Show that $f(0) \neq 1$. (Hint: if ζ and ζ' are complex numbers satisfying monic polynomials in $\mathbb{Z}[X]$, then $\zeta\zeta'$ satisfies a monic polynomial in $\mathbb{Z}[X]$.)

(B) Assume that f is irreducible. Let K be the smallest subfield of $\mathbb{C}$ containing all the roots of f. Let σ be a field automorphism of K such that $\sigma(\alpha) = 3\alpha$. Show that σ has finite order and that $\alpha = 0$.

(20*) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function. Define $g(0) = f(0)$ and $g(x) = \max\{f(y) \mid 0 \leq y \leq x\}$ for $0 < x \leq 1$. Show that g is well-defined and that g is monotone continuous function.

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)