Part B

(11) $\mathcal{T}$ is not Hausdorff; it is the co-finite topology on $\mathbb{C}$. For any polynomial $f$, $f^{-1}(A)$ is a closed set for every finite set $A$, so $f$ is continuous.

(12) $\sum_{n \geq 0} a_n z^n$ is convergent on $\{z \in \mathbb{C} : |z| < 1\}$.

For each real number $\epsilon > 0$, there exists $N_\epsilon$ such that for all $n \geq N_\epsilon$, $|a_n| < \epsilon$. Then we can write

$$|F(z)| \leq C_\epsilon + \sum_{n \geq N_\epsilon} \epsilon |z|^n = C_\epsilon + \frac{\epsilon |z|^{N_\epsilon}}{1 - |z|}$$

for some $C_\epsilon \in \mathbb{R}$ that does not depend on $z$. Hence for $z \in \mathbb{C}$ with $|z| < 1$, $(1 - |z|)|F(z)|$ can be made to take values arbitrarily close to $\epsilon$, for any $\epsilon > 0$, by taking $|z| \to 1$.

By way of contradiction assume that $G = F$. Let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ is a pole of $G$. Let $M$ be the order of the pole at $\zeta$. Write

$$G(z) = \frac{c - M}{(z - \zeta)^M} + \cdots + \frac{c - 1}{(z - \zeta)} + G_1(z)$$

where $G_1(z)$ is an analytic function. As $z \to \zeta$, $(1 - |z|)|G(z)| = |(z - \zeta)G(z)|$ exhibits one of the following behaviours: if $M > 1$, then it approaches infinity; if $M = 1$ (which implies that $c - 1 \neq 0$), it approaches $c - 1 \neq 0$. This is a contradiction.

(13) Assume that $\{|n| : n \in \mathbb{Z}\}$ is bounded. Let $N$ be such that $|n| \leq N$ for every $n \in \mathbb{Z}$. Let $x, y \in \mathbb{R}$. Without loss of generality, $|x| \geq |y|$, and we want to show that $|x + y| \leq |x|$. We have

$$|x + y|^n \leq \sum_{r=0}^{n} \binom{n}{r} |x|^r |y|^{n-r}$$

$$\leq (n + 1)|x|^n$$

for every $n$

Hence $|x + y| \leq N^{\frac{1}{2}}(n + 1)^{\frac{n}{2}}|x|$ for every $n$, so $|x + y| \leq |x|$.

(14) Write $a_n = \left[ \int_0^1 |f(x)|^n dx \right]^{\frac{1}{n}}$. Let $M = \sup\{|f(x)| : 0 \leq x \leq 1\}$. Then $a_n \leq M$ for every $n$, so $\limsup a_n \leq M$. Since $[0, 1]$ is compact, for every $\epsilon > 0$, there exists an interval $I_\epsilon \subseteq [0, 1]$ of positive length such that $M - \epsilon \leq |f(x)| \leq M$ for every $x \in I_\epsilon$. Then

$$a_n \geq \left[ \int_{I_\epsilon} |f(x)|^n dx \right]^{\frac{1}{n}} \geq [(M - \epsilon)^n \cdot \text{length}(I_\epsilon)]^{\frac{1}{n}} = (M - \epsilon)(\text{length}(I_\epsilon))^{\frac{1}{n}}.$$

Hence $\liminf a_n \geq M - \epsilon$ for every $\epsilon > 0$; therefore $\liminf a_n \geq M$, so $\lim a_n = M$.

(15) Without loss of generality, we may assume that $V \neq 0$. Let $M, N \in V$ be non-zero elements. Let $\lambda$ be an eigenvalue of $NM^{-1}$. Then $\det(\lambda M - N) = \det M \det(\lambda n - NM^{-1}) = 0$. However, $(\lambda M - N) \in V$, so $\lambda M - N = 0$. 


(16) If \( p^2 \) divides \( n \), then there is a non-cyclic group \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(n/p^2)\mathbb{Z} \). There are exactly \( pq \) such matrices, and they form a nonabelian group. If \( p_1, \ldots, p_t \) are the distinct prime divisors of \( n \), then \( \phi(n) = \prod_{i=1}^{t} (p_i - 1) \), which is coprime to \( n \), by above.

(17\*) Let \( g \in G \) be of order 2. Then the minimal polynomial of \( g \) divides \( X^2 - 1 \). If \( \text{char} \, F \neq 2 \), then the minimal polynomial of \( g \) is \( X + 1 \) or \( X^2 - 1 \). In either case, \( g \) is diagonalizable and the conjugacy class of \( g \) is determined by the number of \(-1\)s on the diagonal; there must be at least one \(-1\). Hence \( |X| = n \). If \( \text{char} \, F = 2 \), then \( X^2 - 1 = (X - 1)^2 \), which must be the minimal polynomial of \( g \). (It cannot be \( X - 1 \).) Hence \( g \) is not diagonalizable, and the conjugacy class of \( g \) is determined by the number of \( 2 \times 2 \) Jordan blocks (with eigenvalue 1). Hence \( |X| = \lfloor \frac{n}{2} \rfloor \).

(18\*) Consider the sequences \( x_n = \frac{2}{(4n+1)\pi} \) and \( y_n = \frac{1}{n\pi} \). Both converge to 0, but \( f(\frac{2}{(4n+1)\pi}) = 1 \) and \( f(\frac{1}{n\pi}) = 0 \) for every \( n \geq 1 \), so \( f \) does not extend to \( Y \). Let \( h : X \rightarrow X_1, t \mapsto (t, \sin \frac{1}{t}) \). It is a homeomorphism. Since \( X_1 \subseteq [0,1] \times [-1,1], Y_1 \) is compact. Identifying \( X \) with \( X_1 \) using \( h \), we get that \( Y_1 \) is a compactification of \( X \). Further, \( f = \pi_2 \circ h \), where \( \pi_2 \) is the projection \( Y_1 \rightarrow \mathbb{R} \) on to the second component. Note that \( f \) extends to the map \( \pi_2 \) (after identifying \( X \) with \( X_1 \) using \( h \)).

(19\*) Let \( \alpha, 3\alpha, \beta_3, \ldots, \beta_n \) be the roots of \( f \), so \( f(0) = 3\alpha^2 \beta_3 \cdots \beta_n \). If \( f(0) = 1 \), then \( \frac{1}{3} = \alpha^2 \beta_3 \cdots \beta_n \) satisfies a monic irreducible polynomial \( g \in \mathbb{Z}[X] \). Since \( g \) is irreducible in \( \mathbb{Q}[X] \), \( g = (X - \frac{1}{3}) \), which is a contradiction. Hence \( f(0) \neq 1 \).

Note that \( K = \mathbb{Q}(\alpha, 3\alpha, \beta_3, \ldots, \beta_n) \) and that every field automorphism of \( K \) permutes these generators of \( K \). Hence there are at most \( n! \) distinct field automorphisms of \( K \), so \( \sigma \) is of finite order, which we denote by \( m \). Then \( \alpha = \sigma^m(\alpha) = \sigma(\sigma^{m-1}(\alpha)) = 3(3^{m-1}\alpha) = 3^m \alpha \), so \( \alpha = 0 \).

(20\*) Since \([0, x]\) is compact and \( f \) is continuous, one can use max, so \( g \) is well-defined. Note that \( f \) is uniformly continuous, so for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for every \( x, y \) with \( |x - y| < \delta \), \( |f(x) - f(y)| < \epsilon \). Let \( u < v < u + \delta \). Then \( f(u) - \epsilon \leq f(x) \leq f(u) + \epsilon \) for every \( x \in [u, v] \), so \( g(v) \leq g(u) + \epsilon \). On the other hand \( g(v) \geq g(u) \), so \( |g(v) - g(u)| < \epsilon \). Hence \( g \) is continuous.