CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2019

(1) A, B.
(2) D.
(3) A, C.
(4) C, D.
(5) A, B, C.
(6) B, C, D.
(7) C.
(8) C.
(9) C, D.
(10) 0.

Part B

(11) $\mathcal{T}$ is not Hausdorff; it is the co-finite topology on $\mathbb{C}$. For any polynomial f, $f^{-1}(A)$ is a closed set for every finite set A, so f is continuous.

(12) $\sum_{n\geq 0} a_n z^n$ is convergent on $\{z \in \mathbb{C} : |z| < 1\}$.

For each real number $\epsilon > 0$, there exists N_ϵ such that for all $n \geq N_\epsilon$, $|a_n| < \epsilon$. Then we can write

$$|F(z)| \leq C_\epsilon + \sum_{n\geq N_\epsilon} \epsilon |z|^n = C_\epsilon + \frac{\epsilon |z|^{N_\epsilon}}{(1-|z|)}$$

for some $C_\epsilon \in \mathbb{R}$ that does not depend on z. Hence for $z \in \mathbb{C}$ with $|z| < 1$, $(1-|z|)|F(z)|$ can be made to take values arbitrarily close to ϵ, for any $\epsilon > 0$, by taking $|z| \longrightarrow 1$.

By way of contradiction assume that $G = F$. Let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ is a pole of G. Let M be the order of the pole at ζ. Write

$$G(z) = \frac{c_{-M}}{(z-\zeta)^M} + \cdots + \frac{c_{-1}}{(z-\zeta)} + G_1(z)$$

where $G_1(z)$ is an analytic function. As $z \longrightarrow \zeta$, $(1-|z|)|G(z)| = |(z-\zeta)G(z)|$ exhibits one of the following behaviours: if $M > 1$, then it approaches infinity; if $M = 1$ (which implies that $c_{-1} \neq 0$), it approaches $c_{-1} \neq 0$. This is a contradiction.

(13) Assume that $\{|n| : n \in \mathbb{Z}\}$ is bounded. Let N be such that $|n| \leq N$ for every $n \in \mathbb{Z}$. Let $x, y \in \mathbb{R}$. Without loss of generality, $|x| \geq |y|$, and we want to show that $|x+y| \leq |x|$.

$$\begin{aligned} |x+y|^n &\leq \sum_{r=0}^{n} |\binom{n}{r}| |x|^r |y|^{n-r} \\ &\leq (n+1)N|x|^n \quad \text{for every } n \end{aligned}$$

Hence $|x+y| \leq N^{\frac{1}{n}}(n+1)^{\frac{1}{n}}|x|$ for every n, so $|x+y| \leq |x|$.

(14) Write $a_n = \left[\int_0^1 |f(x)|^n \mathrm{d}x\right]^{\frac{1}{n}}$. Let $M = \sup\{|f(x)| : 0 \leq x \leq 1\}$. Then $a_n \leq M$ for every n, so $\limsup a_n \leq M$. Since $[0,1]$ is compact, for every $\epsilon > 0$, there exists an interval $I_\epsilon \subseteq [0,1]$ of positive length such that $M - \epsilon \leq |f(x)| \leq M$ for every $x \in I_\epsilon$. Then

$$a_n \geq \left[\int_{I_\epsilon} |f(x)|^n \mathrm{d}x\right]^{\frac{1}{n}} \geq [(M-\epsilon)^n \cdot \text{length}(I_e)]^{\frac{1}{n}} = (M-\epsilon)(\text{length}(I_e))^{\frac{1}{n}}.$$

Hence $\liminf a_n \geq M - \epsilon$ for every $\epsilon > 0$; therefore $\liminf a_n \geq M$, so $\lim a_n = M$.

(15) Without loss of generality, we may assume that $V \neq 0$. Let $M, N \in V$ be non-zero elements. Let λ be an eigenvalue of NM^{-1}. Then $\det(\lambda M - N) = \det M \det(\lambda I_n - NM^{-1}) = 0$. However, $(\lambda M - N) \in V$, so $\lambda M - N = 0$.

(16) If p^2 divides n, then there is a non-cyclic group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(n/p^2)\mathbb{Z}$. There are exactly pq such matrices, and they form a nonabelian group. If $p_1, \ldots, p_r$ are the distinct prime divisors of n, then $\phi(n) = \prod_{i=1}^{t}(p_i - 1)$, which is coprime to n, by above.

(17*) Let $g \in G$ be of order 2. Then the minimal polynomial of g divides $X^2 - 1$. If $\operatorname{char} F \neq 2$, then the minimal polynomial of g is $X + 1$ or $X^2 - 1$. In either case, g is diagonalizable and the conjugacy class of g is determined by the number of -1s on the diagonal; there must be at least one -1. Hence $|X| = n$. If $\operatorname{char} F = 2$, then $X^2 - 1 = (X - 1)^2$, which must be the minimal polynomial of g. (It cannot be $X - 1$.) Hence g is not diagonalizable, and the conjugacy class of g is determined by the number of 2×2 Jordan blocks (with eigenvalue 1). Hence $|X| = \lfloor \frac{n}{2} \rfloor$.

(18*) Consider the sequences $\frac{2}{(4n+1)\pi}$ and $\frac{1}{n\pi}$. Both converge to 0, but $f(\frac{2}{(4n+1)\pi}) = 1$ and $f(\frac{1}{n\pi}) = 0$ for every $n \geq 1$, so f does not extend to Y. Let $h : X \longrightarrow X_1, t \mapsto (t, \sin \frac{1}{t})$. It is a homeomorphism. Since $X_1 \subseteq [0, 1] \times [-1, 1]$, Y_1 is compact. Identifying X with X_1 using h, we get that Y_1 is a compactification of X. Further, $f = \pi_2 \circ h$, where π_2 is the projection $Y_1 \longrightarrow \mathbb{R}$ on to the second component. Note that f extends to the map π_2 (after identifying X with X_1 using h).

(19*) Let $\alpha, 3\alpha, \beta_3, \ldots, \beta_n$ be the roots of f, so $f(0) = 3\alpha^2\beta_3 \cdots \beta_n$. If $f(0) = 1$, then $\frac{1}{3} = \alpha^2\beta_3 \cdots \beta_n$ satisfies a monic irreducible polynomial $g \in \mathbb{Z}[X]$. Since g is irreducible in $\mathbb{Q}[X]$, $g = (X - \frac{1}{3})$, which is a contradiction. Hence $f(0) \neq 1$.

Note that $K = \mathbb{Q}(\alpha, 3\alpha, \beta_3, \ldots, \beta_n)$ and that every field automorphism of K permutes these generators of K. Hence there are at most $n!$ distinct field automorphisms of K, so σ is of finite order, which we denote by m. Then $\alpha = \sigma^m(\alpha) = \sigma(\sigma^{m-1}(\alpha)) = 3(3^{m-1}\alpha) = 3^m\alpha)$, so $\alpha = 0$.

(20*) Since $[0, x]$ is compact and f is continuous, one can use max, so g is well-defined. Note that f is uniformly continuous, so for every $\epsilon > 0$, there exists $\delta > 0$ such that for every x, y with $|x - y| < \delta$ $|f(x) - f(y)| < \epsilon$. Let $u < v < u + \delta$. Then $f(u) - \epsilon \leq f(x) \leq f(u) + \epsilon$ for every $x \in [u, v]$, so $g(v) \leq g(u) + \epsilon$. On the other hand $g(v) \geq g(u)$, so $|g(v) - g(v)| < \epsilon$. Hence g is continuous.