CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2018

Instructions:

- Enter your *Admit Card Number* here: | M | – | | | | – | | | | |
- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2018

Admit Card Number: M – ☐☐☐ – ☐☐☐☐

Bubble Sheet

Part A

1. (A) (B) (C) (D)
2. (A) (B) (C) (D)
3. (A) (B) (C) (D)
4. (A) (B) (C) (D)
5. (A) (B) (C) (D)
6. (A) (B) (C) (D)
7. (A) (B) (C) (D)
8. (A) (B) (C) (D)
9. (A) (B) (C) (D)
10. ____________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*)**.**

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2018

Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.**

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, of positive real numbers, and of complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}$ are taken with the euclidean topology. When $M_n(\mathbb{R})$ is considered as a topological space, it is identified with $\mathbb{R}^{n^2}$.

Part A

Instructions: Each of the questions 1–9 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let G be a group of order 6. Let $C_1, C_2, \ldots, C_k$ be the distinct conjugacy classes of G. Which of the following sequences of integers are possible values of $(|C_1|, |C_2|, \ldots, |C_k|)$?
(A) $(1,1,1,1,1,1)$;
(B) $(1,5)$;
(C) $(3,3)$;
(D) $(1,2,3)$.

(2) Let $R = \mathbb{F}_2[X]$. Choose the correct statement(s) from below:
(A) R has uncountably many maximal ideals;
(B) Every maximal ideal of R has infinitely many elements;
(C) For all maximal ideals $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is a finite field;
(D) For every integer n, every ideal of R has only finitely many elements of degree $\leq n$.

(3) Which of the following spaces are connected?
(A) $\{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ as a subspace of $\mathbb{R}^2$;
(B) The set of upper triangular matrices as a subspace of $M_n(\mathbb{R})$;
(C) The set of invertible diagonal matrices as a subspace of $M_n(\mathbb{R})$;
(D) $\{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0, z^2 \geq x^2 + y^2\}$ as a subspace of $\mathbb{R}^3$.

(4) Let A be an $n \times n$ nilpotent real matrix A. Define

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .$$

Choose the correct statement(s) from below:

(A) For every real number t, e^{tA} is invertible;

(B) There exists a basis of $\mathbb{R}^n$ such that e^A is upper-triangular;

(C) There exist $B, P \in \mathrm{GL}_n(\mathbb{R})$ such that $B = Pe^AP^{-1}$ and $\mathrm{trace}(B) = 0$;

(D) There exists a basis of $\mathbb{R}^n$ such that A is lower-triangular.

(5) Let $f(w, x, y, z) = wz - xy$. Choose the correct statement(s) from below:

(A) The directional derivative at $(1, 0, 0, 1)$ in the direction (a, b, c, d) is 0 if $a + d = 0$;

(B) The directional derivative at $(1, 0, 0, 1)$ in the direction (a, b, c, d) is 0 only if $a+d = 0$;

(C) The vector $(0, -1, -1, 0)$ is normal to $f^{-1}(1)$ at the point $(1, 0, 0, 1)$;

(D) The set of points (a, b, c, d) where the total derivative of f is zero is finite.

(6) Choose the correct statement(s) from below:

(A) There exists a subfield F of $\mathbb{C}$ such that $F \not\subseteq \mathbb{R}$ and $F \simeq \mathbb{Q}[X]/(2X^3 - 3X^2 + 6)$;

(B) For every irreducible cubic polynomial $f(X) \in \mathbb{Q}[X]$, there exists a subfield F of $\mathbb{C}$ such that $F \not\subseteq \mathbb{R}$ and $F \simeq \mathbb{Q}[X]/f(X)$;

(C) There exists a subfield F of $\mathbb{R}$ such that $F \simeq \mathbb{Q}[X]/(2X^3 - 3X^2 + 6)$;

(D) For every irreducible cubic polynomial $f(X) \in \mathbb{Q}[X]$, there exists a subfield F of $\mathbb{R}$ such that $F \simeq \mathbb{Q}[X]/f(X)$.

(7) For a continuous function $f : [0, 1] \longrightarrow \mathbb{R}$, define $a_n(f) = \int_0^1 x^n f(x)\mathrm{d}x$. Choose the correct statement(s) from below:

(A) The sequence $\{a_n(f)\}$ is bounded for every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$;

(B) The sequence $\{a_n(f)\}$ is Cauchy for every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$;

(C) The sequence $\{a_n(f)\}$ converges to 0 for every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$;

(D) There exists a continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ such that the sequence $\{a_n(f)\}$ is divergent.

(8) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Choose the correct statement(s) from below:
(A) $f(\overline{z})$ is holomorphic;
(B) Suppose that $f(\mathbb{R}) \subseteq \mathbb{R}$. Then $f(\mathbb{R})$ is open in $\mathbb{R}$;
(C) the map $z \mapsto e^{f(z)}$ is holomorphic;
(D) Suppose that $f(\mathbb{C}) \subset \mathbb{R}$. Then $f(A)$ is closed in $\mathbb{C}$ for every closed subset A of $\mathbb{C}$.

(9) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice-differentiable function such that $f(\frac{1}{n}) = 0$ for every positive integer n. Choose the correct statement(s) from below:
(A) $f(0) = 0$;
(B) $f'(0) = 0$;
(C) $f''(0) = 0$;
(D) f is a nonzero polynomial.

Instructions: The answer to Question 10 is an integer. You are required to write the answer in decimal form in the attached bubble-sheet. The question is worth four (4) marks.

(10) Let A be a non-zero 4×4 complex matrix such that $A^2 = 0$. What is the largest possible rank of A?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*).** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

---

(11) A subspace Y of $\mathbb{R}$ is said to be a *retract* of $\mathbb{R}$ if there exists a continuous map $r : \mathbb{R} \longrightarrow Y$ such that $r(y) = y$ for every $y \in Y$.

(A) Show that $[0, 1]$ is a retract of $\mathbb{R}$.

(B) Determine (with appropriate justification) whether every closed subset of $\mathbb{R}$ is a retract of $\mathbb{R}$.

(C) Show that $(0, 1)$ is not a retract of $\mathbb{R}$.

(12) Let N be a positive integer and a_n be a complex number for every $-N \leq n \leq N$. Consider the holomorphic function on $\{z \in \mathbb{C} | z \neq 0\}$ given by

$$F(z) = \sum_{n=-N}^{n=N} a_n z^n.$$

Consider the function f defined on the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\xi)}{\xi - z} d\xi,$$

where Γ is the boundary of the disc, oriented counterclockwise. Write down an expression for f in terms of the coefficients a_n of F.

(13) Let $\phi : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function such that

$$\int_0^1 \phi(t) e^{-at} \mathrm{d}t = 0$$

for every $a \in \mathbb{R}_+$. Show that for every non-negative integer n,

$$\int_0^1 \phi(t) t^n \mathrm{d}t = 0.$$

(14) Let U be a non-empty open subset of $\mathbb{R}$. Suppose that there exists a uniformly continuous homeomorphism $h : U \longrightarrow \mathbb{R}$. Show that $U = \mathbb{R}$.

(15) Let A be 2×2 orthogonal matrix such that $\det(A) = -1$. Show that A represents reflection about a line in $\mathbb{R}^2$.

(16) A subgroup H of a group G is said to be a *characteristic subgroup* if $\sigma(H) = H$ for every group isomorphism $\sigma : G \longrightarrow G$ of G.

(A) Determine all the characteristic subgroups of $(\mathbb{Q}, +)$ (the additive group).

(B) Show that every characteristic subgroup of G is normal in G. Determine whether the converse is true.

(17*) Write V for the space of 3×3 skew-symmetric real matrices.

(A) Show that for $A \in SO_3(\mathbb{R})$ and $M \in V$, $AMA^t \in V$. Write $A \cdot M$ for this action.

(B) Let $\Phi : \mathbb{R}^3 \longrightarrow V$ be the map

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{bmatrix}.$$

With the usual action of $SO_3(\mathbb{R})$ on $\mathbb{R}^3$ and the above action on V, show that $\Phi(Av) = A \cdot \Phi(v)$ for every $A \in SO_3(\mathbb{R})$ and $v \in \mathbb{R}^3$.

(C) Show that there does not exist $M \in V$, $M \neq 0$ such that for every $A \in SO_3(\mathbb{R})$, $A \cdot M$ belongs to the span of M.

(18*) Let $m > 1$ be an integer and consider the following equivalence relation on $\mathbb{C} \setminus \{0\}$: $z_1 \sim z_2$ if $z_1 = z_2 e^{\frac{2\pi \imath a}{m}}$ for some $a \in \mathbb{Z}$. Write X for the set of equivalence classes and $\pi : \mathbb{C} \setminus \{0\} \longrightarrow X$ for the map that takes z to its equivalence class. Define a topology on X by setting $U \subseteq X$ to be open if and only if $\pi^{-1}(U)$ is open in the euclidean topology of $\mathbb{C} \setminus \{0\}$. Determine (with appropriate justification) whether X is compact.

(19*) Let $\Bbbk$ be a field, n a positive integer and G a *finite* subgroup of $\mathrm{GL}_n(\Bbbk)$ such that $|G| > 1$. Further assume that every $g \in G$ is upper-triangular and all the diagonal entries of g are 1.

(A) Show that $\operatorname{char} \Bbbk > 0$. (Hint: consider the minimal polynomials of elements of G.)

(B) Show that the order of g is a power of $\operatorname{char} \Bbbk$, for every $g \in G$.

(C) Show that the centre of G has at least two elements.

(20*) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function. Determine (with appropriate justification) the following limit:

$$\lim_{n \longrightarrow \infty} \int_0^1 n x^n f(x) \mathrm{d}x.$$

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)