CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
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Part A

(1) A, D.
(2) B, C, D.
(3) B, D.
(4) A, B, D.
(5) A, B, D.
(6) A, C, D.
(7) A, B, C.
(8) C, D.
(9) A, B, C.
(10) 2.

Part B

(11) (A) Consider $r : \mathbb{R} \longrightarrow [0, 1]$

$$r(x) = \begin{cases} 0, & r < 0; \\ x, & r \in [0, 1]; \\ 1, & r > 1. \end{cases}$$

(B) No. Every retract Y of $\mathbb{R}$ must be connected because the map r is continuous.

(C) Every retract Y of $\mathbb{R}$ is closed. To see this, consider $\phi : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}, x \mapsto (r(x), x)$. Then $Y = \phi^{-1}(\text{diagonal})$. Since $\mathbb{R}$ is Hausdorff, the diagonal is closed, and so is Y.

(12) Write

$$g_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\xi^n}{\xi - z} d\xi.$$

Note that

$$g_n(z) = \begin{cases} 1, & n = 0, z = 0 \\ 0, & n \neq 0, z = 0 \\ z^n, & z \neq 0 \end{cases}$$

Hence

$$f(z) = \sum_{n=-N}^{n=N} a_n g_n(z) = \begin{cases} a_0, & z = 0; \\ F(z), & z \neq 0. \end{cases}$$

(13) Write $F_n = (-1)^n \frac{\int_0^1 \phi(t) t^n \mathrm{d}t}{n!}$. Then

$$\begin{aligned} 0 = \int_0^1 \phi(t) e^{-at} \mathrm{d}t &= \int_0^1 \phi(t) \left(\sum_{n=0}^{\infty} \frac{(-at)^n}{n!} \right) \mathrm{d}t \\ &= \lim_{N \longrightarrow \infty} \int_0^1 \phi(t) \left(\sum_{n=0}^{N} \frac{(-at)^n}{n!} \right) \mathrm{d}t \\ &= \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} \int_0^1 \phi(t) \left(\frac{(-at)^n}{n!} \right) \mathrm{d}t \\ &= \sum_{n=0}^{\infty} F_n a^n \end{aligned}$$

Since $\sum_{n=0}^{\infty} F_n a^n = 0$ for every $a \in \mathbb{R}_+$, we see that $F_n = 0$ for every $n \geq 0$.

(14) By way of contradiction assume that $U \subsetneq \mathbb{R}$. Then, since U is open, $U \subsetneq \overline{U}$. Pick $x \in \overline{U} \smallsetminus U$ and a sequence $\{x_n\} \subseteq U$ converging to x. Since h is uniformly continuous, $\{h(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, it converges to $y \in \mathbb{R}$. Hence $\{x_n\}$ converges go $h^{-1}(y) \in U$, a contradiction.

(15) Since $\det A = -1$, the characteristic polynomial of A is of the form $X^2 + bX - 1$ for some $b \in \mathbb{R}$, so A has real eigenvalues, λ_1, λ_2. Let v_i be an eigenvector for λ_i, $i = 1, 2$. For $i = 1, 2$, $\lambda_i^2 v_i^t v_i = \lambda_i v_i^t \lambda_i v_i = v_i^t A^t A v_i = v_i^t v_i$, so λ_i is 1 or -1. Without loss of generality, $\lambda_1 = 1$ and $\lambda_2 = -1$. Then A gives a reflection about the line spanned by v_1 sending v_2 to $-v_2$.

(16) (A) For any G, 0 is a characteristic subgroup. Let $0 \neq H \subseteq \mathbb{Q}$ be a characteristic subgroup. Let $0 \neq x \in H$ and $y \in \mathbb{Q}$. Then the map $r \mapsto ry/x$ is an automorphism of $\mathbb{Q}$, and it takes x to y. Hence $y \in H$, so $H = \mathbb{Q}$. Hence 0 and $\mathbb{Q}$ are the only characteristic subgroups of $\mathbb{Q}$.

(B) For any $g \in G$, the map $G \longrightarrow G$, $g_1 \mapsto g g_1 g^{-1}$ is an isomprphism, so, for every characteristic subgroup H of G, $gHg^{-1} = H$. Hence H is normal. The converse is false: take $H = \mathbb{Z}$ inside $G = \mathbb{Q}$.

(17*) (A) $(A^t M A)^t = A^t M^t A = -A^t M A$.

(B) Check directly with $v = e_1, e_2, e_3$ after noting tha the action of $SO_E(\mathbb{R})$ and Φ are $\Bbbk$-linear. Also use the fact that the entries $A^t = A^{-1}$ are the signed minors of A, coming from the adjoint of A.

(C) Using (B) it is enough to show that there does not exist a nonzero $v \in \mathbb{R}^3$ whose span is stable under the action of SO_3. This is true: if $v, w \in \mathbb{R}^3$ are non-zero vectors of the same length, there exists $A \in SO_3$ such that $Av = w$.

(18*) For positive integers n, write $U_n = \{z \in \mathbb{C} \smallsetminus \{0\} : |z| < n\}$. These form an open cover of $\mathbb{C} \smallsetminus \{0\}$. Note that for every $z_1 \sim z_2$, $|z_1| = |z_2|$, so for every n, $\pi^{-1}(\pi(U_n)) = U_n$. Hence $\pi(U_n), n \geq 1$ is an open cover of X. This does not have a finite sub-cover since the open cover $U_n, n \geq 1$ does not have a finite sub-cover.

(19*) First note that the minimal polynomial of g divides $X^{|G|} - 1$. (A) If $\operatorname{char} \Bbbk = 0$ then for every g, g has distinct eigenvalues and hence is diagonalizable, so $g = 1$, contradicting the hypothesis that $|G| > 1$. (B) Let $p = \operatorname{char} \Bbbk$. Let $g \in G$ and write its order as $p^e m$ with $m = 1$ or $m > 1$ and $p \nmid m$. The the minimal polynomial of g^{p^e} is $X^m - 1$ which has distinct roots, so, again, by the above argument, $g^{p^e} = 1$, so $m = 1$. (C) Hence G is a p-group. Use class equation.

(20*) The limit is $f(1)$. This is true for $x^k, k \geq 0$, and hence also for polynomials. By Weierstrass' theorem, it is true for all continuous functions.