CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2017

Instructions:

- Enter your *Admit Card Number* here: | M | – | | | | – | | | | |
- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

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Admit Card Number: **M** – ☐☐☐ – ☐☐☐☐

Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. Ⓐ Ⓑ Ⓒ Ⓓ
9. Ⓐ Ⓑ Ⓒ Ⓓ
10. ____________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*).

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Number of correct answers in Part A:	
Marks in Part A:	

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Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*).

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, and of complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1. When considered as a topological space, $\mathbb{R}^n$ is taken with the euclidean topology.

Part A

Instructions: Each of the questions 1–9 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let G be a finite subgroup of $\mathrm{GL}_n(\Bbbk)$ where $\Bbbk$ is an algebraically closed field. Choose the correct statement(s) from below:
(A) Every element of G is diagonalizable;
(B) Every element of G is diagonalizable if $\Bbbk$ is an algebraic closure of $\mathbb{Q}$;
(C) Every element of G is diagonalizable if $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$;
(D) There exists a basis of $\Bbbk^n$ with respect to which every element of G is a diagonal matrix.

(2) Consider the ideal $I := (ux, uy, vx, uv)$ in the polynomial ring $\mathbb{Q}[u, v, x, y]$, where u, v, x, y are indeterminates. Choose the correct statement(s) from below:
(A) Every prime ideal containing I contains the ideal (x, y);
(B) Every prime ideal containing I contains the ideal (x, y) or the ideal (u, v);
(C) Every maximal ideal containing I contains the ideal (u, v);
(D) Every maximal ideal containing I contains the ideal (u, v, x, y).

(3) Let f be an irreducible cubic polynomial over $\mathbb{Q}$ with at most one real root and $\Bbbk$ the smallest subfield of $\mathbb{C}$ containing the roots of f. Choose the correct statement(s) from below:
(A) $\sigma(K) \subseteq K$ where σ denotes complex conjugation;
(B) $[K : \mathbb{Q}]$ is an even number;
(C) $[(K \cap \mathbb{R}) : \mathbb{Q}]$ is an even number;
(D) K is uncountable.

(4) For a positive integer n, let S_n denote the permutation group on n symbols. Choose the correct statement(s) from below:

(A) For every positive integer n and for every m with $1 \leq m \leq n$, S_n has a cyclic subgroup of order m;

(B) For every positive integer n and for every m with $n < m < n!$, S_n has a cyclic subgroup of order m;

(C) There exist positive integers n and m with $n < m < n!$ such that S_n has a cyclic subgroup of order m;

(D) For every positive integer n and for every group G of order n, G is isomorphic to a subgroup of S_n.

(5) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$, both taken with the subspace topology of $\mathbb{R}^2$. Choose the correct statement(s) from below:

(A) Every continuous function from A to $\mathbb{R}$ has bounded image;

(B) There exists a non-constant continuous function from B to $\mathbb{N}$ (in the subspace topology of $\mathbb{R}$);

(C) For every surjective continuous function from $A \cup B$ to a topological space X, X has at most two connected components;

(D) B is homeomorphic to the unit circle.

(6) Let (X, d) be a metric space. Choose the correct statement(s) from below:

(A) There exists a metric $\tilde{d}$ on X such that d and $\tilde{d}$ define the same topology and such that $\tilde{d}$ is bounded (i.e., there exists a real number M such that $\tilde{d}(x, y) < M$ for all $x, y \in X$.);

(B) Every closed subset of X that is bounded with respect to d is compact;

(C) X is connected;

(D) For every $x \in X$, there exists $y \in X$ such that $d(x, y)$ is a non-zero rational number.

(7) Which of the following are equivalence relations on $\mathbb{R}$?

(A) $a \sim b$ if and only if $|a - b| \leq 25$;

(B) $a \sim b$ if and only if $a - b$ is rational;

(C) $a \sim b$ if and only if $a - b$ is irrational;

(D) $a \sim b$ if and only if $f(a) = f(b)$ for every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(8) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be two differentiable functions such that $f(x+1, y) = f(x, y+1) = f(x, y)$ and $g(x+1, y) = g(x, y+1) = g(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Choose the correct statement(s) from below:
(A) f is uniformly continuous;
(B) f is bounded;
(C) The function $(f, g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is differentiable;
(D) If $\partial f/\partial x = \partial g/\partial y$ and $\partial f/\partial y = -\partial g/\partial x$, then the function $\mathbb{C} \longrightarrow \mathbb{C}$ sending $(x + \imath y) \longrightarrow f(x, y) + \imath g(x, y)$ (with $x, y \in \mathbb{R}$) is constant.

(9) Consider the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}.$$

Choose the correct statement(s) from below:
(A) There exists $(a, b) \in \mathbb{R}^2$ satisfying the above equation;
(B) There exists $(a, b) \in \mathbb{C}^2$ satisfying the above equation;
(C) There exists $(a, b) \in \mathbb{C}^2$ with $a = b$ satisfying the above equation;
(D) There exists $(a, b) \in (\mathbb{F}_3)^2$ with $a = b$ satisfying the above equation.

Instructions: The answer to Question 10 is an integer. You are required to write the answer in decimal form in the attached bubble-sheet. The question is worth <u>four</u> (4) marks.

(10) Let $p = (0, 0), q = (0, 1), r = (\imath, 0)$ be points of $\mathbb{C}^2$. What is the dimension of the $\mathbb{C}$-vector space

$$\{f(X, Y) \in \mathbb{C}[X, Y] \mid \deg f \leq 2 \text{ and } f(p) = f(q) = f(r) = 0\},$$

where by $\deg f$, we mean the total degree of the polynomial f?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Let (X, τ) be a topological space and $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ a continuous function where $X \times X$ has the product topology and $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers, with the subspace topology of the usual topology of $\mathbb{R}$. Assume that $d^{-1}(0) = \{(x, x) \mid x \in X\}$, and that $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. Show the following:

(A) (X, τ) is Hausdorff.

(B) The sets $B_{x,\epsilon} := \{y \in X \mid d(x, y) < \epsilon\}$, $0 < \epsilon \in \mathbb{R}$ is the basis for a topology τ' on X.

(C) τ' is coarser than τ (i.e., every set open in τ' is open in τ).

(12) (A) Let f be an entire function such that $|f(z)| \leq |z|$. Show that f is a polynomial of degree ≤ 1.

(B) Let Γ be a closed differentiable contour oriented counterclockwise and let

$$\int_\Gamma \overline{z}\ \mathrm{d}z = A.$$

What is the integral

$$\int_\Gamma (x + y)\ \mathrm{d}z$$

(where x and y, respectively, are the real and imaginary parts of z) in terms of A?

(13) Let f_n, f be real-valued functions on $[0, 1]$ with f continuous. Suppose that for all convergent sequences $\{x_n : n \geq 1\} \subseteq [0, 1]$ with $x = \lim_{n\to\infty} x_n$ one has

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Show that f_n converges to f uniformly.

(14) (A) Show that for any positive rational number r, the sequence $\{\frac{\log n}{n^r} : n \geq 1\}$ is bounded.

(B) Show that the series

$$\sum_{n\geq 10} \frac{(\log n)^2(\log\log n)}{n^2}$$

is convergent.

(15) For a group G, let $\mathrm{Aut}(G)$ denote the group of group automorphisms of G. (The group operation of $\mathrm{Aut}(G)$ is composition.) Let p be prime number. Show that the multiplicative group $\mathbb{F}_p \setminus \{0\}$ is isomorphic to $\mathrm{Aut}((\mathbb{F}_p, +))$ under the map $a \mapsto [b \mapsto ab]$ ($a \in \mathbb{F}_p \setminus \{0\}$, $b \in \mathbb{F}_p$).

(16) Let $\Bbbk$ be a field, X an indeterminate and $R = \Bbbk[X]/(X^7 - 1)$. Determine the set

$$\{\dim_{\Bbbk} R/\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal in } R\}$$

in the following three cases: $\Bbbk = \mathbb{Q}$; $\Bbbk = \mathbb{C}$; $\Bbbk$ is a field of characteristic 7.

(17*) For a 3×3 matrix A, say that a point p on the unit sphere centred at the origin in $\mathbb{R}^3$ is a *pole* of A if $Ap = p$. Denote by SO_3 the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of all the orthogonal matrices with determinant 1.

(A) Show that if $A \in \mathrm{SO}_3$, then A has a pole.

(B) Let G be a subgroup of SO_3. Show that G acts on the set

$$\{p \in \mathbb{S}^2 \mid p \text{ is a pole for some matrix } A \in G\}.$$

(18*) Let $f : X \longrightarrow Y$ be a continuous surjective map such that for every closed $A \subseteq X$, $f(A)$ is closed in Y. Show that if Y and all the fibres $f^{-1}(y)$, $y \in Y$ are compact, then X is compact. Show that if Y is Hausdorff and X is compact, then Y and the all fibres $f^{-1}(y)$, $y \in Y$ are compact.

(19*) Let $\Bbbk$ be an algebraically closed uncountable field and $\mathfrak{m}$ a maximal ideal in the polynomial ring $R := \Bbbk[x_1, \ldots, x_n]$ in the indeterminates $x_1, \ldots, x_n$. Show that the composite map $\Bbbk \longrightarrow R \longrightarrow R/\mathfrak{m}$ is a field isomorphism. You may use without proof the following fact from linear algebra: If a vector space has a countable spanning set, it cannot have a linearly independent uncountable set in it. (Hint: If t is transcendental over $\Bbbk$, then consider the set $\{\frac{1}{t-\alpha} \mid \alpha \in \Bbbk\}$.)

(20*) Prove that for every $z \in \mathbb{C}$, the series $\sum_{n=1}^{\infty} \frac{\sin(z/n)}{n}$ converges. For $z \in \mathbb{C}$, let $f(z) = \sum_{n=1}^{\infty} \frac{\sin(z/n)}{n}$. Prove that f is entire.

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)