CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2017
Solutions

Part A

(1) B.
(2) B.
(3) A, B.
(4) A, C, D.
(5) C
(6) A.
(7) B, D.
(8) A, B, C, D.
(9) B, D.
(10) 3.

Part B

(11) (A) X is Hausdorff since the diagonal is closed.
(B) For all x, y, $d(x, y) \leq d(x, x) + d(y, x) = d(y, x)$, so $d(x, y) = d(y, x)$; hence d is a metric. In particular, τ' is the metric topology given by d.
(C) Identify $B_{x_0,\epsilon}$ with $d^{-1}([0, \epsilon)) \cap (\{x_0\} \times X)$. Since $d^{-1}([0, \epsilon))$ is open in $X \times X$, we see that $d^{-1}([0, \epsilon)) \cap (\{x_0\} \times X)$ is open in the subspace topology of $(\{x_0\} \times X)$ induced from the product topology of $X \times X$, which is the same as the topology τ on X. Hence $B_{x_0,\epsilon}$ is open in the topology τ, so τ' is coarser than τ.

(12) (A) f has a power-series expansion around 0 that converges everywhere on $\mathbb{C}$. Since $|f(0)| \leq 0$, $f(0) = 0$, so $f(z)/z$ is entire, and $|f(z)/z| \leq 1$. Hence, by Liouville's theorem, $f(z)/z = C$ for some $C \in \mathbb{C}$, i.e., $f(z) = Cz$.
(B)

$$\int_\Gamma z \, \mathrm{d}z = 0$$

and use the fact that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2\imath}$.

(13) By way of contradiction, assume that there exist $\epsilon > 0$, an increasing sequence $n_1 < n_2 < \cdots$ of integers and $x_1, x_2, \ldots \in [0, 1]$ such that $|f_{n_k}(x_k) - f(x_k)| > \epsilon$ for every $k \geq 1$. Let $x_{k_i}, i \geq 1$ be a convergent subsequence, converging to $y \in [0, 1]$. Construct a new sequence y_j as follows:

$$y_j = \begin{cases} x_{k_i}, & \text{if} j = n_{k_i} \\ y, & \text{otherwise.} \end{cases}$$

Then the sequence $f_j(y_j)$ does not converge to $f(y)$, a contradiction.

(14) (A) If r is a positive rational number, then $\log(x) < x^r$ for all real numbers $x \gg 0$.
(B) The two sequences below are bounded and the series is convergent:

$$\frac{\log n}{n^{0.1}}; \qquad \frac{\log \log n}{n^{0.1}}; \qquad \sum \frac{1}{n^{3/2}}.$$

(15) Write $\phi_a : \mathbb{F}_p \longrightarrow \mathbb{F}_p$ for the map $b \mapsto ab$. We check the following:
(A) For each $a \in \mathbb{F}_p \smallsetminus \{0\}$, ϕ_a is a group automorphism.
(B) The map $a \mapsto \phi_a$ is a group homomorphism: $1 \mapsto \mathrm{id}_{\mathbb{F}_p}$; $\phi_{a'a}(b) = a'ab = (\phi_{a'} \circ \phi_a)(b)$.
(C) The map $a \mapsto \phi_a$ is injective: Indeed if $\phi_a = \mathrm{id}_{\mathbb{F}_p}$ then $a = 1$.

(D) The map $a \mapsto \phi_a$ is surjective: Let ϕ be any group automorphism of $(\mathbb{F}_p, +)$, which is a cyclic group, generated by 1. Then ϕ is determined by $\phi(1)$. Since ϕ is an automorphism, $\phi(1) \neq 0$. Hence $\phi(b) = \phi(1)b$ for every $b \in \mathbb{F}_p$. Therefore $\phi = \phi_{\phi(1)}$.

(E) A bijective group homomorphism is a group isomorphism.

(16) In each of the three cases, if $\mathfrak{m}$ is a maximal ideal of R, then it is generated by (the residue class of) an irreducible polynomial dividing $X^7 - 1$. Further, if $f(X)$ is an irreducible polynomial of degree d, then $\dim_{\Bbbk} \Bbbk[X]/f(X) = d$.

$\underline{\Bbbk = \mathbb{Q}}$: The irreducible factors of $X^7 - 1$ are $(X-1)$ and $(X^6 + X^5 + \cdots + 1)$. (To see that $(X^6 + X^5 + \cdots + 1)$ is irreducible over $\mathbb{Q}$, write it as $((Y+1)^6 + (Y+1)^5 + \cdots + 1)$ (where $Y = X - 1$) and apply the Eisenstein criterion.) Hence the dimensions are 1 and 6.

$\underline{\Bbbk = \mathbb{C}}$: Every irreducible polynomial is linear, so the dimension is 1.

$\underline{\operatorname{char} \Bbbk = 7}$: $X^7 - 1 = (X-1)^7$, so the dimension is 1.

(17*) A has a pole if and only if 1 is an eigenvalue. If $A \in \mathrm{SO}_3$, then its real eigenvalues are ± 1, and its determinant is 1. Hence if all the eigenvalues are real, then at least one eigenvalue is 1. If it has exactly one real eigenvalue λ_1, then $1 = \lambda_1(a + \imath b)(a - \imath b)$, so $\lambda_1 > 0$, i.e., $\lambda_1 = 1$. For the second part, we need to show that for $p \in \mathbb{S}^2$ if $Ap = p$ for some $A \in G$ then for every $B \in G$, Bp is a pole for some $C \in G$. Take $C = BAB^{-1}$.

(18*) First suppose that Y and the all fibres $f^{-1}(y)$, $y \in Y$ are compact. We prove a 'Tube lemma': Let $y \in Y$; if U is an open neighbourbood of $f^{-1}(y)$, then there exists an open neighbourbood V of y such that $f^{-1}(V) \subseteq U$. Proof of lemma: $X \setminus U$ is closed, so $f(X \setminus U)$ is closed. Since $f^{-1}(y) \subseteq U$, $y \notin f(X \setminus U)$. Let $V_y = Y \setminus f(X \setminus U)$. One can check immediately that $f^{-1}(V_y) \subseteq U$, finishing the proof of the lemma.

Let $U_\lambda, \lambda \in \Lambda$ be an open cover of X. For each $y \in Y$, we see, using the above lemma and the fact that $f^{-1}(y)$ is compact, that there is a finite subset $\Lambda_y \subset \Lambda$ and an open neighbourbood V_y of y such that $f^{-1}(y) \subseteq f^{-1}(V_y) \subseteq \bigcup_{\lambda \in \Lambda_y} U_\lambda$. Since Y is compact, there exist $y_1, \ldots, y_n \in Y$ such that $Y = V_{y_1} \cup \cdots \cup V_{y_n}$. Thus $X = f^{-1}(Y) = f^{-1}(V_{y_1}) \cup \cdots \cup f^{-1}(V_{y_n}) \subseteq \bigcup_{i=1}^{n} \bigcup_{\lambda \in \Lambda_{y_i}} U_\lambda$, so the open cover $U_\lambda, \lambda \in \Lambda$ has a finite subcover. Hence X is compact.

Conversely, assume that X is compact and Y is Hausdorff. Then Y is compact. Let $y \in Y$. Then $\{y\}$ is closed in Y, so $f^{-1}(y)$ is closed in X, and hence compact, as X is compact.

(19*) The images in $F := R/\mathfrak{m}$ of the monomials in $x_1, \ldots, x_n$ form a countable spanning set of F over $\Bbbk$. If $t \in F$ is transcendental over $\Bbbk$, then $\{\frac{1}{t-\alpha} \mid \alpha \in \Bbbk\}$ is linearly independent over $\Bbbk$, which is not possible, so $F/\Bbbk$ is algebraic. Hence $F \simeq \Bbbk$.

(20*) Let $R > 0$ be given and $B(0, R)$ be the open unit ball of radius R. By the mean value inequality, $|\sin(\frac{z}{n}) - \sin(0)| \leq \sup_{w \in B(0,R)} |\cos(w)||\frac{z}{n} - 0|$ for every $z \in B(0, R)$. Hence $\sum_{n=1}^{\infty} \frac{\sin \frac{z}{n}}{n}$ is dominated by the series $\sum_{n=1}^{\infty} |z|/n^2$ as $\cos(w)$ is bounded on $B(0, R)$.

Therefore the series converges uniformly on compact sets. Hence the series $\sum_{n=1}^{\infty} \sin(\frac{z}{n})/n$ defines a holomorphic function. It is a standard fact that if a sequence of holomorphic functions converges uniformly on compact sets then the limit is a holomorphic function.