Part A

(1) B.
(2) B.
(3) A, B.
(4) A, C, D.
(5) C
(6) A.
(7) B, D.
(8) A, B, C, D.
(9) B, D.
(10) 3.

Part B

(11) (A) X is Hausdorff since the diagonal is closed.
(B) For all $x, y, d(x, y) \leq d(x, x) + d(y, x) = d(y, x)$, so $d(x, y) = d(y, x)$; hence $d$ is a metric. In particular, $\tau'$ is the metric topology given by $d$.
(C) Identify $B_{x, \epsilon}$ with $d^{-1}([0, \epsilon]) \cap (\{x\} \times X)$. Since $d^{-1}([0, \epsilon])$ is open in $X \times X$, we see that $d^{-1}([0, \epsilon]) \cap (\{x\} \times X)$ is open in the subspace topology of $(\{x\} \times X)$ induced from the product topology of $X \times X$, which is the same as the topology $\tau$ on $X$. Hence $B_{x, \epsilon}$ is open in the topology $\tau$, so $\tau'$ is coarser than $\tau$.

(12) (A) $f$ has a power-series expansion around 0 that converges everywhere on $\mathbb{C}$. Since $|f(0)| \leq 0$, $f(0) = 0$, so $f(z)/z$ is entire, and $|f(z)/z| \leq 1$. Hence, by Liouville's theorem, $f(z)/z = C$ for some $C \in \mathbb{C}$, i.e., $f(z) = Cz$.
(B) $\int_{\Gamma} z \, dz = 0$
and use the fact that $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$.

(13) By way of contradiction, assume that there exist $\epsilon > 0$, an increasing sequence $n_1 < n_2 < \cdots$ of integers and $x_1, x_2, \ldots \in [0, 1]$ such that $|f_{n_k}(x_k) - f(x_k)| > \epsilon$ for every $k \geq 1$. Let $x_k, i \geq 1$ be a convergent subsequence, converging to $y \in [0, 1]$. Construct a new sequence $y_j$ as follows:

$$y_j = \begin{cases} x_{k_i}, & \text{if } j = n_{k_i} \\ y, & \text{otherwise.} \end{cases}$$

Then the sequence $f_j(y_j)$ does not converge to $f(y)$, a contradiction.

(14) (A) If $r$ is a positive rational number, then $\log(x) < x^r$ for all real numbers $x \gg 0$.
(B) The two sequences below are bounded and the series is convergent:

$$\frac{\log n}{n^{0.1}}; \frac{\log \log n}{n^{0.1}}; \sum \frac{1}{n^{3/2}}.$$

(15) Write $\phi_a : \mathbb{F}_p \rightarrow \mathbb{F}_p$ for the map $b \mapsto ab$. We check the following:
(A) For each $a \in \mathbb{F}_p \setminus \{0\}$, $\phi_a$ is a group automorphism.
(B) The map $a \mapsto \phi_a$ is a group homomorphism: $1 \mapsto \text{id}_{\mathbb{F}_p}$; $\phi_{a'}(b) = a'ab = (\phi_a \circ \phi_{a'})(b)$.
(C) The map $a \mapsto \phi_a$ is injective: Indeed if $\phi_a = \text{id}_{\mathbb{F}_p}$ then $a = 1$. 

(D) The map \( a \mapsto \phi_a \) is surjective: Let \( \phi \) be any group automorphism of \((\mathbb{F}_p, +)\), which is a cyclic group, generated by 1. Then \( \phi \) is determined by \( \phi(1) \). Since \( \phi \) is an automorphism, \( \phi(1) \neq 0 \). Hence \( \phi(b) = \phi(1)b \) for every \( b \in \mathbb{F}_p \). Therefore \( \phi = \phi_{\phi(1)} \).

(E) A bijective group homomorphism is a group isomorphism.

(16) In each of the three cases, if \( \mathfrak{m} \) is a maximal ideal of \( R \), then it is generated by \( (\text{the residue class of}) \) an irreducible polynomial dividing \( X^7 - 1 \). Further, if \( f(X) \) is an irreducible polynomial of degree \( d \), then \( \text{dim}_{\mathbb{K}}[X]/f(X) = d \).

\[ k = \mathbb{Q}: \text{The irreducible factors of } X^7 - 1 \text{ are } (X - 1) \text{ and } (X^6 + X^5 + \cdots + 1). \text{(To see that } (X^6 + X^5 + \cdots + 1) \text{ is irreducible over } \mathbb{Q}, \text{write it as } ((Y + 1)^6 + Y^5 + \cdots + 1) \text{ where } Y = X - 1) \text{ and apply the Eisenstein criterion.) Hence the dimensions are } 1 \text{ and } 6. \]

\[ k = \mathbb{C}: \text{Every irreducible polynomial is linear, so the dimension is } 1. \]

\[ \text{char } k = 7: \text{ } X^7 - 1 = (X - 1)^7, \text{ so the dimension is } 1. \]

(17*) \( A \) has a pole if and only if 1 is an eigenvalue. If \( A \in \text{SO}_3 \), then its real eigenvalues are \( \pm 1 \), and its determinant is 1. Hence if all the eigenvalues are real, then at least one eigenvalue is 1. If it has exactly one real eigenvalue \( \lambda_1 \), then \( 1 = \lambda_1(a + b)(a - ib) \), so \( \lambda_1 > 0 \), i.e., \( \lambda_1 = 1 \). For the second part, we need to show that for \( p \in \mathbb{S}^2 \) if \( Ap = p \) for some \( A \in G \), then \( f \in G \). Take \( C = BAB^{-1} \).

(18*) First suppose that \( Y \) and the all fibres \( f^{-1}(y), y \in Y \) are compact. We prove a `Tube lemma`: Let \( y \in Y \); if \( U \) is an open neighbourhood of \( f^{-1}(y) \), then there exists an open neighbourhood \( V \) of \( y \) such that \( f^{-1}(V) \subseteq U \). Proof of lemma: \( X \setminus U \) is closed, so \( f(X \setminus U) \) is closed. Since \( f^{-1}(y) \subseteq U \), \( y \not\in f(X \setminus U) \). Let \( V_y = Y \setminus f(X \setminus U) \). One can check immediately that \( f^{-1}(V_y) \subseteq U \), finishing the proof of the lemma.

Let \( U_{\lambda}, \lambda \in \Lambda \) be an open cover of \( X \). For each \( y \in Y \), we see, using the above lemma and the fact that \( f^{-1}(y) \) is compact, that there is a finite subset \( \Lambda_y \subseteq \Lambda \) and an open neighbourhood \( V_y \) of \( y \) such that \( f^{-1}(y) \subseteq f^{-1}(V_y) \subseteq \bigcup_{\lambda \in \Lambda_y} U_{\lambda} \). Since \( Y \) is compact, there exist \( y_1, \ldots, y_n \in Y \) such that \( Y = V_{y_1} \cup \cdots \cup V_{y_n} \). Thus \( X = f^{-1}(Y) = f^{-1}(V_{y_1}) \cup \cdots \cup f^{-1}(V_{y_n}) \subseteq \bigcup_{i=1}^n \bigcup_{\lambda \in \Lambda_{y_i}} U_{\lambda} \), so the open cover \( U_{\lambda}, \lambda \in \Lambda \) has a finite subcover.

Hence \( X \) is compact.

Conversely, assume that \( X \) is compact and \( Y \) is Hausdorff. Then \( Y \) is compact. Let \( y \in Y \). Then \( \{y\} \) is closed in \( Y \), so \( f^{-1}(y) \) is closed in \( X \), and hence compact, as \( X \) is compact.

(19*) The images in \( F := R/\mathfrak{m} \) of the monomials in \( x_1, \ldots, x_n \) form a countable spanning set of \( F \) over \( k \). If \( t \in F \) is transcendental over \( k \), then \( \{\frac{1}{\alpha} | \alpha \in k\} \) is linearly independent over \( k \), which is not possible, so \( F/k \) is algebraic. Hence \( F \cong k \).

(20*) Let \( R > 0 \) be given and \( B(0, R) \) be the open unit ball of radius \( R \). By the mean value inequality, \( |\sin(\frac{z}{n}) - \sin(0)| \leq \sup_{w \in B(0, R)} |\cos(w)||\frac{z}{n} - 0| \) for every \( z \in B(0, R) \). Hence \( \sum_{n=1}^{\infty} \frac{\sin \frac{z}{n}}{n} \) is dominated by the series \( \sum_{n=1}^{\infty} |z|/n^2 \) as \( \cos(w) \) is bounded on \( B(0, R) \).

Therefore the series converges uniformly on compact sets. Hence the series \( \sum_{n=1}^{\infty} \frac{\sin(\frac{z}{n})}{n} \) defines a holomorphic function. It is a standard fact that if a sequence of holomorphic functions converges uniformly on compact sets then the limit is a holomorphic function.