CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2016
Solutions

Part A

(1) A,D.
(2) A.
(3) A,C.
(4) B, C.
(5) A.
(6) B.
(7) A, B, C, D.
(8) A, C, D.
(9) 1.
(10) 1.

Part B

(11) In polar coordinates, the region U is given by $U' := (1,2) \times [0, 2\pi)$. Write p' and q' for the points in U' corresponding to p and q respectively. Let $\gamma' : [0,1] \longrightarrow U'$ be given by $t \mapsto t \cdot p' + (1-t) \cdot q'$. This is continuous, and is such that $\gamma'(0) = p'$ and $\gamma'(1) = q'$ and such that γ' is differentiable on $(0,1)$. Let f be the map (converting polar coordinates to Cartesian coordinates),

$$\mathbb{R} \times [0, 2\pi) \longrightarrow \mathbb{R}^2, (r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

Let $\gamma = f \circ \gamma'$. Since f is differentiable, γ has the desired properties.

(12) By way of contradiction, suppose that IJ is a prime ideal. In an integral domain, a product of two non-zero ideals is non-zero, so IJ is a non-zero prime ideal. In a PID, every non-zero prime ideal is maximal, so IJ is maximal. Since $IJ \subseteq I \cap J$, we conclude that $I = IJ = J$. Let a be a generator for I and J; hence IJ is generated by a^2. Therefore we see that $a \in (a^2)$. Write $a = ba^2$, so $a(1 - ba) = 0$. Since $a \neq 0$, $ba = 1$, i.e., a is a unit, so I is not a proper ideal, a contradiction.

(13) Write $z = x + \imath y$ with $x, y \in \mathbb{R}$ and $f(z) = u(x, y) + \imath v(x, y)$, where u and v are maps from $\mathbb{R}^2$ to $\mathbb{R}$. Then $v(0,t) = 0 = v(t, 0)$ for every $t \in \mathbb{R}$. Therefore

$$\frac{\partial v}{\partial x}(0,0) = 0 = \frac{\partial v}{\partial y}(0,0).$$

Since f is entire,

$$\frac{\partial u}{\partial x}(0,0) = 0 = \frac{\partial u}{\partial y}(0,0).$$

Therefore $f'(0) = 0$.

(14) Let $\epsilon > \delta$, and consider the closed ball

$$B := \{y \in \mathbb{R}^n | d(x,y) \leq \epsilon\}.$$

There exists $y \in A$ such that $d(x,y) \leq \epsilon$ so

$$\delta = \inf\{d(x,y) \mid y \in A \cap B\}.$$

Since $A \cap B$ is closed and bounded, it is compact. Consider the function $f : A \cap B \longrightarrow \mathbb{R}$, $y \mapsto d(x,y)$. It is continuous, so it attains its infimum and supremum, i.e., there exists $y \in A \cap B \subseteq A$ such that $\delta = d(x,y)$.

(15) (A): Fix a basis $v_1, \dots, v_n$ of V. Let $n_1, \dots, n_d$ be such that $T^{n_i} v_i = v_i$. Let $n = \operatorname{lcm}\{n_i \mid 1 \le i \le d\}$. Then $T^n(v) = v$ for every $v \in V$, so, over $\mathbb{C}$, the minimal polynomial of T has distinct roots. For (B), take $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{F}_2$.

(16) $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$, since the minimal polynomial of ω is $X^2 + X + 1$. A basis of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$ is $\{1, \omega\}$. The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $X^3 - 2$, so $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is a basis of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$. Therefore $\{1, \sqrt[3]{2}, \sqrt[3]{4}, \omega, \omega\sqrt[3]{2}, \omega\sqrt[3]{4}\}$ span F as $\mathbb{Q}$-vector-space. Since $X^3 - 2$ is irreducible over $\mathbb{Q}(\omega)$, $[F : \mathbb{Q}(\omega)] = 3$, and, therefore $[F : \mathbb{Q}] = 6$. Hence $\{1, \sqrt[3]{2}, \sqrt[3]{4}, \omega, \omega\sqrt[3]{2}, \omega\sqrt[3]{4}\}$ is a $\mathbb{Q}$-basis for F. Since $\omega^2 = -(1 + \omega)$ and $\omega^3 = 1$, we see that the matrix of μ with respect to the above basis (in the given order) is

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

(17*) Suppose that G is not dense. Let $l := \inf\{x \in G \mid x > 0\}$. We first show that $l > 0$ and that $l \in G$. If $l = 0$, then there exists a small open neighbourhood U of $l = 0$ that contains non-zero elements of G; Now, for every $g \in G$, $\{g + u \mid u \in U\}$ is an open neighbourhood of g that contains elements of G other than g, so G is dense; hence $l > 0$. Now, if $l \notin G$, then there exists $\epsilon > 0$ such that $(l, l + \epsilon) \subseteq G$. We may assume, without loss of generality, that $\epsilon < l$. Let $x, y \in (l, l + \epsilon)$, with $x < y$. Then $y - x \in G$ and $0 < y - x < \epsilon$. This contradicts the choice of l, so $l \in G$.

Now let $g \in G$, and let n be the largest integer such that $nl \le g < (n + 1)l$. Hence $0 \le g - nl < l$, so by the minimality of l, $g = nl$, i.e., $G = \mathbb{Z} \cdot l$.

(18*) Let $\phi : X \longrightarrow \mathbb{C}$ be the constant function taking the value 1. Then for every $g \in G$, $(g \cdot \phi)(x) = \phi(g^{-1}(x)) = 1 = \phi(x)$ for every $x \in X$, so $g \cdot \phi = \phi$ for every $g \in G$. Let

$$F' := \{f \in F \mid \sum_{x \in X} f(x) = 0\}.$$

It is a subspace of F. To show that $F = F' \oplus \mathbb{C}\langle\phi\rangle$, we need to show that $F' \cap \mathbb{C}\langle\phi\rangle = 0$ and that $F = F' + \mathbb{C}\langle\phi\rangle$. If $\alpha\phi \in F'$ for some $\alpha \in \mathbb{C}$, then $0 = \sum_{x \in X} (\alpha\phi)(x) = \alpha|X|$, so $\alpha = 0$, i.e., $F' \cap \mathbb{C}\langle\phi\rangle = 0$. Let $f \in F$. Set $\alpha = \frac{1}{|X|} \sum_{x \in X} f(x)$. Then $f - \alpha\phi \in F'$, so $F = F' + \mathbb{C}\langle\phi\rangle$.

(19*) (A) Write $A = (a_{ij})$ and $B = (b_{ij})$. If $a_{ij} > 1$ then for every k, $b_{jk} = 0$, for otherwise, the (ik)th entry of AB would be greater than 1. However, if there exists j such that for every k $b_{jk} = 0$, then B is not invertible. Hence $a_{ij} \in \{0, 1\}$ for every i, j. Similarly $b_{ij} \in \{0, 1\}$ for every i, j. Now suppose that $a_{ij} = a_{ik} = 1$ with $j < k$. Then exactly one of b_{ji}, b_{ki} is 1 and for $l \ne i$, $b_{jl} = b_{kl} = 0$. Therefore either the jth row or the kth row of B is zero, a contradiction. Hence, for every i, there is a unique k_i such that $a_{ik_i} = 1$. Therefore A is a permutation matrix. Since $B = A^{-1}$, B is a permutation matrix.

(B) Let $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the linear transformation given by $e_i \mapsto Ae_i$, where the e_i are the standard basis of $\mathbb{C}^n$. Let $v \in \mathbb{C}^n$ be a non-zero vector that is not an eigenvector of A (and of T). (Such a vector exists by the hypothesis on A.) Then v and Av are linearly independent. Extend it to a basis $v, Av, v_3, \dots, v_n$ of $\mathbb{C}^n$. In this basis, T is given by a matrix B that satisfies $B_{1,1} = 0$. Since A and B represent the same linear transformation (in two different bases), they are similar to each other.

(20*) Define $f : S^1 \times \{1, -1\} \longrightarrow S^1$ by setting $f(w, 1) = \operatorname{Sqrt}(w)$ and $f(w, -1) = -\operatorname{Sqrt}(w)$. It is continuous. Surjectivity: Let $w \in S^1$. If $\operatorname{Sqrt}(w^2) = w$, then $f(w, 1) = w$; otherwise, $\operatorname{Sqrt}(w^2) = -w$, and hence $f(w, -1) = w$. Injectivity: If $f(w_1, 1) = f(w_2, -1)$, then $w_1 = (f(w_1, 1))^2 = (f(w_2, -1))^2 = w_2$; however $f(w, 1) \ne f(w, -1)$ for any w. Therefore there do not exist w_1, w_2 such that $f(w_1, 1) = f(w_2, -1)$. If $f(w_1, 1) = f(w_2, 1)$ then $w_1 = (f(w_1, 1))^2 = (f(w_2, 1))^2 = w_2$; similarly if $f(w_1, -1) = f(w_2, -1)$ then $w_1 = w_2$. Since $S^1 \times \{1, -1\}$ and S^1 are Hausdorff and compact, f is a homeomorphism, a contradiction, since $S^1 \times \{1, -1\}$ is not connected, while S^1 is.