2015 MSc/PhD Mathematics Solutions

Part A

(1). B, D.
(2). B.
(3). A, B, C, D.
(4). A, C.
(5). C, D.
(6). A, C.
(7). B, C.
(8). 2.
(9). 0.
(10). 0.

Part B

(11). After changing coordinates, we may assume that f vanishes in an open neighbourhood of the origin. Hence

$$\left(\frac{\partial^n}{\partial x^i \partial y^{n-i}} f\right)(0,0) = 0$$

for every $n \geq 0$ and $0 \leq i \leq n$. Therefore all the coefficients of f are zero, so $f = 0$.

(12). (a). Let $v \neq 0 \in \ker A$, so $(I_n + A)v = v$; hence 1 is an eigenvalue of $(I_n + A)$. Conversely, suppose that $(I_n + A)v = \lambda v$ for some nonzero v. If $Av = 0$, then $v = \lambda v$, so $\lambda = 1$. Otherwise, multiplying on the left by A, we see that $Av = \lambda(Av)$; hence, again, $\lambda = 1$.

(b). $A^2 + 1 = 0$, so the minimal polynomial of A divides $t^2 + 1$ which has distinct roots. Hence the minimal polynomial of A has distinct roots, so A is diagonalizable.

(13). For $r \in \mathbb{R}, r > 0$, let $C_r := \{z \in \mathbb{C} : |z| = r\}$, $U_r := \{z \in \mathbb{C} : |z| < r\}$ and $\epsilon_r = \min\{|f(z)| : z \in C_r\}$. As f it nonconstant, the origin is an isolated zero of f. Choose $r > 0$ such that the origin is the only zero in $C_r \cup U_r$. As the set $\{z : |f(z)| < \epsilon_r\}$ intersects U_r but not C_r, it must be in U_r, by the second condition on f. Hence the origin is the only zero of f. Now letting $r \longrightarrow \infty$, we see that for all $z \notin U_r$, $|f(z)| \geq \epsilon_r$. Hence f has a pole at infinity, i.e., there exists a non-negative integer m such that $z^m f(z^{-1})$ is analytic at $z = 0$. Suppose that f is a power-series expression $\sum_{i=n}^{\infty} a_i z^i$. Since $z^m f(z^{-1})$ is analytic at $z = 0$, we see that $a_i = 0$ for all $i \geq m$. Hence f is a polynomial that vanishes exactly at $z = 0$. Thus $f = cz^n$ for some positive interger n and non-zero $c \in \mathbb{C}$.

(14). For each positive integer k, using a), choose N_k such that

$$|b_k - a_{m,k}| < \frac{\epsilon}{2} \; \forall m \geq N_k.$$

Let $m_k = N_1 + \cdots + N_k$. Then m_k is increasing. Now, using b), choose a positive integer K, such that

$$|a_{m_k,k} - 1| < \frac{\epsilon}{2} \; \forall k \geq K.$$

By triangle inequality

$$|b_k - 1| < \epsilon \; \forall k \geq K.$$

(15). May assume that f is homogeneous of degree d. Induct on d. If x^d appears in f, then so does y^d (with the same coefficient. Hence we may write $f = \alpha(x^d + y^d) + xyf_1(x, y)$, where α could be zero. Note that $f_1(x, y) = f_1(y, x)$, so by induction, there exists g_1 such that $f_1(x, y) = g_1(x + y, xy)$. Write $x^d + y^d = (x + y)^d - xyf_2(x, y)$; $f_2(x, y) = f_2(y, x)$, so by induction, there exists g_2 such that $f_2(x, y) = g_2(x + y, xy)$.

(16). Note that for every $a \in [0, 1]$ and for every open subset U of X containing $f^{-1}(a)$, there is an open neighbourhood W (for example, take $W = [0, 1] \setminus f(X \setminus U)$) of a such that $f^{-1}(W) \subseteq U$. Now let $U_i, i \in I$ be an open covering of X. For every $a \in [0, 1]$, there exists a finite subset I_a of I such that $f^{-1}(a) \subseteq \bigcup_{i \in I_a} U_i$. Let W_a be an open neighbourhood of a such that $f^{-1}(W_a) \subseteq \bigcup_{i \in I_a} U_i$. There exist finitely many $a_1, \ldots, a_n$ such that $[0,1] = \bigcup_{j=1}^n W_{a_j}$. Then $X = \bigcup_{j=1}^n f^{-1}(W_{a_j}) = \bigcup_{j=1}^n \bigcup_{i \in I_{a_j}} U_i$. This is a finite subcovering of the given open covering.

(17*). For each such P, let R_P be the smallest subring of $\mathbb{Q}$ that contains $\{\frac{1}{p} | p \in P\}$. If $P \neq Q$, then $R_P \neq R_Q$. The set of all the primes is countably infinite, and the power-set of a countably infinite set is uncountable. Hence $\mathbb{Q}$ has uncountably many subrings.

(18*). Let $|x| \leq M$. Since $\sin x \leq x$,

$$f(x) = \sum_{n \geq 1} \frac{\sin \frac{x}{n}}{n} \leq M \sum_{n \geq 1} \frac{1}{n^2} < \infty,$$

and hence f is well-defined.

Also the same argument shows, if we set $f_N = \sum_{N \geq n \geq 1} \frac{\sin \frac{x}{n}}{n}$, then f_N converges uniformly to f as $N \mapsto \infty$ in the interval $(-M, M)$, and hence f is continuous. Since M is arbitrary f is continuous everywhere.

Again let $|x| \leq M$ and we set $g = \sum_{n \geq 1} \frac{\cos \frac{x}{n}}{n^2}$ Since $\cos \frac{x}{n} \leq 1$, g is well-defined, and $f_N' = \sum_{N \geq n \geq 1} \frac{\cos \frac{x}{n}}{n^2}$ uniformly converges to g, by the same arguments. Since f_N converges uniformly to f and since f_N' converges uniformly to g in the interval $(-M, M)$, $f' = g$. Since M is arbitrary, f is differentiable everywhere.

(19*). By the usual computation, $p^{m(m-1)/2}$ is the largest power of p dividing the order of $\mathrm{GL}_m(\mathbb{F}_p)$. Also, one may check directly that U is a subgroup of $\mathrm{GL}_m(\mathbb{F}_p)$ and its order is $p^{m(m-1)/2}$. Therefore, U is a Sylow p-subgroup of $\mathrm{GL}_m(\mathbb{F}_p)$. Now, by Sylow's Theorem 2 and its Corollary, there exists $A \in \mathrm{GL}_n(\mathbb{F}_p)$ such that $AGA^{-1} \subset U$.

(20*).

Let $A \in M_{m \times n}(\mathbb{C})$ be the matrix that consists of 1s on the diagonal on rows and columns $1, \ldots, k$ and 0s elsewhere. Then the space of all matrices of rank k inside $M_{m \times n}(\mathbb{C})$ is

$$X := \{B_1 A B_2 : B_1 \in \mathrm{GL}_m(\mathbb{C}); B_2 \in \mathrm{GL}_n(\mathbb{C})\};$$

we need to show that this space is connected. Fix $B_2 \in \mathrm{GL}_n(\mathbb{C})$. Then the space

$$X_{B_2} := \{B_1 A B_2 : B_1 \in \mathrm{GL}_m(\mathbb{C})\}$$

is connected since it is a continuous image of the connected space $\mathrm{GL}_m(\mathbb{C})$. Similarly, the space

$$Y := \{A B_2 : B_2 \in \mathrm{GL}_n(\mathbb{C})\}$$

is connected. Since

$$X = Y \cup \bigcup_{B_2 \in \mathrm{GL}_n(\mathbb{C})} X_{B_2}$$

and each X_{B_2} intersects Y, we see that X is connected.