CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2014

Instructions:

- Enter your *Registration Number* here **CMIPG ID–** ☐☐☐☐☐☐
  OR here **M–** ☐☐☐☐
- Enter the name of the city where you write this test: ☐☐☐☐☐☐☐☐☐☐☐☐
- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. If you are applying for the PhD Mathematics programme (possibly in addition to other programmes), at least two (2) should be from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Part A	
Part B	
Total	

Further remarks:

This page is intentionally left blank.

CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2014

- *Registration Number*: **CMIPG ID**– | | | | | | | OR **M**– | | | | |
- City where you write this test: | | | | | | | | | | | | |

Bubble Sheet

Part A

1. (A) (B) (C) (D)
2. (A) (B) (C) (D)
3. (A) (B) (C) (D)
4. (A) (B) (C) (D)
5. (A) (B) (C) (D)
6. (A) (B) (C)
7. (A) (B) (C)
8. (A) (B) (C) (D)
9. __________
10. __________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. If you are applying for the PhD Mathematics programme (possibly in addition to other programmes) then you should answer at least two (2) from among the starred questions (17^*)–(20^*).

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

This page is intentionally left blank.

CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics

MSc/PhD Entrance Examination

15 May 2014

Important: Questions in Part A will be used for **screening**. Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account while making the final decision.

For qualifying for the PhD interview, you should answer at least two (2) from among the starred questions (17*)–(20*) in Part B.

Notation: $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of the natural numbers, of the rational numbers, of the real numbers, and of the complex numbers. For a field F, $M_{m\times n}(F)$ stands for the set of $m \times n$ matrices over F. We treat $M_{m\times n}(\mathbb{R})$ and $M_{m\times n}(\mathbb{C})$ as metric spaces with the metric $d(A,B) = \sqrt{\sum_{i,j} |a_{ij} - b_{ij}|^2}$ where $A = (a_{ij})$ and $B = (b_{ij})$.

Part A

Instructions: Each of the questions 1–8 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Which of the following statement(s) is/are true?
(A) f is bounded.
(B) f is bounded if it is continuous.
(C) f is differentiable if it is continuous.
(D) f is uniformly continuous if it is continuous.

(2) Let $W \subset \mathbb{R}^n$ be a linear subspace of dimension at most $n-1$. Which of the following statement(s) is/are true?
(A) W is nowhere dense.
(B) W is closed.
(C) $\mathbb{R}^n \setminus W$ is connected.
(D) $\mathbb{R}^n \setminus W$ is not connected.

(3) Let G be a finite group. An element $a \in G$ is called a *square* if there exists $x \in G$ such that $x^2 = a$. Which of the following statement(s) is/are true?
(A) If $a, b \in G$ are not squares, ab is a square.
(B) Suppose that G is cyclic. Then if $a, b \in G$ are not squares, ab is a square.
(C) G has a normal subgroup.
(D) If every proper subgroup of G is cyclic then G is cyclic.

(4) Let $A \in M_{m\times n}(\mathbb{R})$ and let $b_0 \in \mathbb{R}^m$. Suppose the system of equations $Ax = b_0$ has a unique solution. Which of the following statement(s) is/are true?
(A) $Ax = b$ has a solution for every $b \in \mathbb{R}^m$.
(B) If $Ax = b$ has a solution then it is unique.
(C) $Ax = 0$ has a unique solution.
(D) A has rank m.

(5) Let $A \in M_{n\times n}(\mathbb{C})$. Which of the following statement(s) is/are true?
  (A) There exists $B \in M_{n\times n}(\mathbb{C})$ such that $B^2 = A$.
  (B) A is diagonalizable.
  (C) There exists an invertible matrix P such that PAP^{-1} is upper-triangular.
  (D) A has an eigenvalue.

(6) Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Which of the following statement(s) is/are true?
  (A) Consider f as a function $(f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that for $i = 1, 2$, both $\frac{\partial f_i}{\partial X}$ and $\frac{\partial f_i}{\partial Y}$ exist and are continuous. Then f is entire.
  (B) Assume that f is entire and $|f(z)| < 1$ for all $z \in \mathbb{C}$. Then f is constant.
  (C) Assume that f is entire and $\mathrm{Im}(f(z)) > 0$ for all $z \in \mathbb{C}$. Then f is constant.

(7) Let $\mathcal{C}(\mathbb{R})$ be the $\mathbb{R}$-vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let a_1, a_2, a_3 be distinct real numbers. For $i = 1, 2, 3$, let $f_i \in \mathcal{C}(\mathbb{R})$ be the function $f_i(t) = e^{a_i t}$. Which of the following statement(s) is/are true?
  (A) f_1, f_2 and f_3 are linearly independent.
  (B) f_1, f_2 and f_3 are linearly dependent.
  (C) f_1, f_2 and f_3 form a basis of $\mathcal{C}(\mathbb{R})$.

(8) Which of the following statement(s) is/are true?
  (A) The series $\sum_{n=1}^{\infty} \mathrm{e}^{-n^2}$ converges.
  (B) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
  (C) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges absolutely.
  (D) The series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$.

Instructions: The answers to questions 9 and 10 are integers. You are required to write the answers in decimal form in the attached bubble-sheet. Every question is worth four (4) marks.

(9) What is the dimension of the ring $\mathbb{Q}[x]/((x+1)^2)$ as a $\mathbb{Q}$-vector space?

(10) Evaluate $\lim_{n\to\infty} \left[\frac{\pi \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)}{n} \right]$.

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **For qualifying for the PhD interview, you should answer at least two (2) from among starred questions (17*)–(20*).** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

---

(11) Show that the set of rank two matrices in $M_{2\times 3}(\mathbb{R})$ is open.

(12) (A) Let F be a finite field extension of $\mathbb{Q}$. Show that any field homomorphism $\phi : F \to F$ is an isomorphism. (Note that $\phi(1) = 1$ by definition.)

(B) Let F be a finite field whose characteristic is not 2. Let $F^\times$ denote the multiplicative group of nonzero elements of F. An element $a \in F^\times$ is called a *square* if there exists $x \in F^\times$ such that $x^2 = a$. Show that exactly half the elements $F^\times$ are squares.

(13) Let $n \in \mathbb{N}$. Show that the determinant map $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is infinitely differentiable and compute the total derivative $d(\det)$ at every point $A \in M_{n\times n}(\mathbb{R})$. Find a necessary and sufficient condition on the rank of A for $d(\det) = 0$ at A.

(14) Let $a_i, i \in \mathbb{R}$ be non-negative real numbers such that $\sup \left\{ \sum_{i\in F} a_i \mid F \subseteq \mathbb{R} \text{ a finite subset} \right\}$ is finite. Show that $a_i = 0$ except for countably many $i \in \mathbb{R}$. Give an example to show that 'countably' cannot be replaced by 'finite'. (Hint: consider $F_n := \{i \mid a_i \geq \frac{1}{n}\}$.)

(15) Let G be a finite group of order $2n$ for some integer n. Consider the map $\phi : G \to G$ given by $\phi(a) = a^2$. Show that ϕ is not surjective.

(16) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.
(A) Construct a sequence $\{z_n\}$ in $\mathbb{C}$ such that $|z_n| \to \infty$ and $e^{z_n} \to 1$.
(B) Show that the function $g(z) = f(e^z)$ is not a polynomial.

(17*) For $F = \mathbb{R}$ and $F = \mathbb{C}$, let $O_n(F) = \{A \in M_{n\times n}(F) \mid AA^t = I_n\}$.
(A) Show that $O_n(\mathbb{R})$ is compact.
(B) Is $O_n(\mathbb{R})$ connected? Justify.
(C) Is $O_n(\mathbb{C})$ compact? Justify.

(18*) Let Ω be a region in $\mathbb{C}$. Let $\{a_n\}$ be a sequence of nonzero elements in Ω such that $a_n \to 0$ as $n \to \infty$. Let $\{b_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \frac{b_n}{a_n^k} = 0$ for every nonnegative integer k. Suppose that $f : \Omega \to \mathbb{C}$ is an entire function such that $f(a_n) = b_n$ for all n. Show that $b_n = 0$ for every n.

(19*) Let G be a finite group of order n and let H be a subgroup of G of order m. Assume that $(\frac{n}{m})! < 2n$. Show that G is not simple, that is: G has a nontrivial proper normal subgroup. (Hint: Think along the lines of Cayley's theorem.)

(20*) Let

$$\mathcal{C}_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}, \lim_{x\to\infty} |f(x)| = 0 \text{ and } \lim_{x\to-\infty} |f(x)| = 0\}, \text{ and}$$

$$\mathcal{C}_0^\infty(\mathbb{R}) = \{f \in \mathcal{C}_0(\mathbb{R}) \mid f \text{ is infinitely differentiable}\}.$$

Let $\phi \in \mathcal{C}_0(\mathbb{R})$. For $f \in \mathcal{C}_0(\mathbb{R})$, define $\phi^*(f) = f \circ \phi$.
(A) Show that $\phi^*(f) \in \mathcal{C}_0(\mathbb{R})$ if $f \in \mathcal{C}_0(\mathbb{R})$.
(B) If $\phi^*(\mathcal{C}_0^\infty(\mathbb{R})) \subseteq \mathcal{C}_0^\infty(\mathbb{R})$, then show that ϕ is infinitely differentiable.

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)