MSC/PHD MATHEMATICS 2014 SOLUTIONS

Part A

1. B, D.
2. A, B.
3. B.
4. B, C.
5. C, D.
6. B, C.
7. A.
8. A, B, D.
9. **2**.
10. **2**.

Part B

(11) The set of rank 2 matrices in $M_{2\times3}(\mathbb{R})$ is open: Consider the map $f : M_{2\times3}(\mathbb{R}) \to \mathbb{R}^3$ given by sending a matrix to the triple of its 2×2 minors. This is a continuous map. The set of rank 2 matrices is the inverse image of the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i \neq 0 \text{ for some } 1 \leq i \leq 3\}$. This set is open in $\mathbb{R}^3$, hence the set of rank 2 matrices is open in $M_{2\times3}(\mathbb{R})$.

(12) (A) The kernel of ϕ is a proper ideal in F. Hence it is zero, as there are no nonzero proper ideals in a field. Since $\phi(1) = 1$, using properties of a field homomorphism we conclude that $\phi(r) = r$ for every $r \in \mathbb{Q}$. Now if $r \in \mathbb{Q}$ and $x \in F$, then $\phi(rx) = \phi(r)\phi(x) = r\phi(x)$. Thus ϕ is a homomorphism of vector spaces over $\mathbb{Q}$. Since ϕ is injective and dimension of F over $\mathbb{Q}$ is finite, it follows that ϕ is also surjective. Thus ϕ is a field isomorphism.

(B) Consider the map $\phi : F^\times \to F^\times$ defined by $\phi(x) = x^2$. Then ϕ is a group homomorphism and the set of squares in $F^\times$ is the image of ϕ. The kernel of ϕ is the set of all $x \in F^\times$ such that $x^2 = 1$. Since F is a field this equation has at most 2 solutions. Further since the characteristic of F is different from 2, it has exactly two solutions. Hence the kernel of ϕ contains two elements and we have an isomorphism $F^\times/\ker(\phi) \cong \mathrm{im}(\phi)$. Hence the cardinality of $\mathrm{im}(\phi)$ is half of the cardinality of $F^\times$.

(13) (a) For $A \in M_n(\mathbb{C})$, $\det(A)$ is a polynomial in the entires of A, so all the multiple partial derivatives exist and are continuous.

(b) Let $A_{ij} \in M_{n-1}(\mathbb{C})$ be the matrix with ith row and jth column removed from A. Then the total derivative $d(\det)$ is the matrix in $M_n(\mathbb{C})$ whose (i,j)-th entry is $(-1)^{i+j}\det(A_{ij})$.

(c) $d(\det)$=0 if and only if $\det(A_{ij}) = 0$ for all i, j. This is equivalent to $\mathrm{rank}(A_{ij}) \leq n-2$ for all i, j. We prove that this is equivalent to $\mathrm{rank}(A) \leq n-2$.

Suppose $\mathrm{rank}(A) \leq n-2$. This means the rows (or columns) of A have at most $n-2$ linearly independent vectors. After removing a row and column there will be at most $n-2$ linearly independent vectors. Hence $\mathrm{rank}(A_{ij}) \leq n-2$ for all i, j. Conversely if $\mathrm{rank}(A) > n-2$, the rows of A have at least $n-1$ linearly independent vectors. Choose $n-1$ linearly independent row vectors and form a matrix of size $(n-1) \times n$. Then $n-1$ columns of this matrix are linearly independent. So we can remove one column and the remaining $(n-1) \times (n-1)$ matrix will still have $(n-1)$ linearly independent vectors. So there exits a pair (i,j) such that $\mathrm{rank}(A_{ij}) = n-1$.

Hence $d(\det)$(A)=0 if and only if $\mathrm{rank}(A) \leq n-2$.

(14) Let $F_0 = \{i \mid a_i \neq 0\}$ then $F_0 = \cup_{n=1}^{\infty} F_n$. Each F_n is a finite set, otherwise

$$\sup\{\sum_{i \in F} a_i : F \subseteq \mathbb{R} \text{ finite subset}\} \geq \sup\{\sum_{i \in F} a_i : F \subseteq F_n \text{ finite subset}\} \geq \frac{N}{n} \; \forall N \in \mathbb{N}.$$

We can not replace countability by finiteness, since any convergent series with infinitely many non-zero entries will satisfy the condition. Take for example $a_i = \frac{1}{i^2}$ for $i \in \mathbb{N}$ and $a_i = 0$ for $i \in \mathbb{R} \setminus \mathbb{N}$.

(15) Since the order of G is divisible by 2, G has an element x of order 2, by Cauchy's theorem. Since $x \neq 1$ and $\phi(x) = 1$, ϕ is not injective. As ϕ is a function from a finite set to itself, it can not be surjective as well.

(16) Let $z_n = 2\pi i n$. Then $|z_n| \to \infty$ and $e^{z_n} \to 1$. Suppose $g(z) = f(e^z)$ is a non-constant polynomial. We have $g(2\pi n i) = f(1)$ for all $n \in \mathbb{N}$. So $g - f(1)$ has infinitely many zeros and hence $g = f(1)$. Thus f is equal to the constant $f(1)$ on $\mathbb{C} \setminus \{0\}$, which is the range of the function e^z. It follows by continuity that f is constant on all of $\mathbb{C}$.

(17*) (A) $O_n(\mathbb{R})$ is compact : Consider the map $f : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ given by $f(A) = AA^t - I_n$. This map is continuous and $O_n(\mathbb{R})$ is closed since it is equal to $f^{-1}(0)$. $O_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ is bounded: the condition $AA^t = I_n$ expressed in terms of the entries of A gives (for example when $n = 2$) the following: let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The we get $a^2 + b^2 = 1 = c^2 + d^2$, and $ac + bd = 0$. This implies $|a|, |b|, |c|, |d| \leq 1$.

(B) Consider $\det : O_n(\mathbb{R}) \to \{1, -1\}$. This is continuous and surjective. This shows that $O_n(\mathbb{R})$ is disconnected.

(C) $O_n(\mathbb{C})$ is not compact, since it is not bounded as a subset of $M_{n\times n}(\mathbb{C})$. The conditions expressed above for a, b, c, d when the entries are complex numbers show this. For example, fixing $a, |a| \gg 0$, there are solutions for b such that $a^2 + b^2 = 1$. Similarly for c, d.

(18*) f is given entire. Expand it as a power series in a neighbourhood of 0. $f = \sum_{n=0}^{\infty} r_n.z^n$. We are given: $f(a_j) = \sum_{n=0}^{\infty} r_n.a_j^n = b_j$ for all j. Taking limits as $j \to \infty$ on either side and using uniform convergence, we see that $f(0) = \lim_{j\to\infty} b_j$. But we are given $\lim_{j\to\infty} \frac{b_j}{a_j^k} = 0$ for all $k \geq 0$. Taking $k = 0$, we get $\lim_{j\to\infty} b_j = 0$, hence $r_0 = f(0) = 0$.

Now write $f(z) = \sum_{n=1}^{\infty} r_n.z^n$. Proceeding as before and using $\lim_{j\to\infty} \frac{b_j}{a_j^k} = 0$ for $k = 1$, we get $r_1 = 0$. Similarly $r_j = 0$ for all j and hence $f = 0$ in a neighbourhood of 0. Since f is entire, we get $f = 0$ everywhere and $b_n = 0$ for all n.

(19*) Following Cayley's theorem, use the action of G on the set of left cosets G/H. This gives a homomorphism $f : G \to S_N$, where $N = \frac{n}{m} = |G/H|$.

Since $|S_N| = N! < 2n$, it implies that $\frac{|S_N|}{2} < n = |G|$. If f is injective, then by the above inequality, f is actually an isomorphism. In this case, G is not simple because S_N is not, as it has the alternating group A_N as a normal subgroup. On the other hand, if f is not injective, it has a non-trivial kernel $K \subset G$. K is a proper nonzero normal subgroup of G, hence again G is not simple.

(20*) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|\phi(x)| \to \infty$ as $|x| \to \infty$. Then by the formula for the limit of composition of two maps, one gets $f \circ \phi \in C_0(\mathbb{R})$ for every $f \in C_0(\mathbb{R})$.

Suppose that $f \circ \phi$ is infinitely differentiable for every f in $C_0^\infty(\mathbb{R})$. Let $a \in \mathbb{R}$. The image $\phi[a-1, a+1]$ is a compact interval. Choose $f \in C_0^\infty(\mathbb{R})$ such that $f = 1$ on $\phi[a-1, a+1]$. Then $f \circ \phi = \phi$ on $(a-1, a+1)$. Thus ϕ agrees with the infinitely differentiable function $f \circ \phi$ on the neighbourhood $(a-1, a+1)$ of a. Hence ϕ is infinitely differentiable in a neighbourhood of a. As a is an arbitrary point of $\mathbb{R}$, ϕ is differentiable.